The University of Sydney

MATH1002 Linear Algebra

Summer School **Exercise Sheet 3** 2012

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Geometric definition of dot product: If $\mathbf{v}$ and $\mathbf{w}$ are vectors and θ is the angle between them, then

$$\mathbf{v}\cdot\mathbf{w} \;=\; |\mathbf{v}||\mathbf{w}|\cos\theta\,,$$

so that

$$\cos\theta \;=\; \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\,.$$

(ii) Algebraic definition of dot product: If $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ and $\mathbf{w} = d\,\mathbf{i} + e\,\mathbf{j} + f\,\mathbf{k}$ then

$$\mathbf{v}\cdot\mathbf{w} \;=\; ad + be + cf\,.$$

(iii) The angle between two vectors is zero or acute if their dot product is positive. The angle is obtuse or 180° if the dot product is negative. Two vectors are mutually perpendicular if the dot product is zero.

(iv) Cauchy-Schwarz Inequality: $|\mathbf{v}\cdot\mathbf{w}| \;\le\; |\mathbf{v}||\mathbf{w}|\,.$

(v) Commutativity of dot product: $\mathbf{v}\cdot\mathbf{w} \;=\; \mathbf{w}\cdot\mathbf{v}\,.$

(vi) Distributivity of dot over plus: $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} \;=\; \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}\,.$

(vii) If $\mathbf{v}$ is any vector then $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$, so $|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}\,.$

(viii) If $\mathbf{v}$ and $\mathbf{w}$ are vectors and λ is a scalar then $(\lambda\mathbf{v})\cdot\mathbf{w} = \lambda(\mathbf{v}\cdot\mathbf{w}) = \mathbf{v}\cdot(\lambda\mathbf{w})\,.$

(ix) The *vector projection* of $\mathbf{v}$ in the direction of $\mathbf{w}$ is $\dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$, which is the best approximation of $\mathbf{v}$ using a scalar multiple of $\mathbf{w}$.

(x) The *scalar component* of $\mathbf{v}$ in the direction of $\mathbf{w}$ is $\dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|}$, which is plus or minus the magnitude of the vector projection (minus in the case that the angle is obtuse or 180°).

(xi) The *vector component of* $\mathbf{v}$ *orthogonal to* $\mathbf{w}$ is the difference between $\mathbf{v}$ and its vector projection, which is

$$\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\,.$$

Preparatory Exercises (answers below):

1. Given that

$$\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}\,, \qquad \mathbf{v} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\,, \qquad \mathbf{w} = 3\mathbf{i} - \mathbf{k}\,,$$

find

(i) $\mathbf{u}\cdot\mathbf{v}$ (ii) $\mathbf{u}\cdot\mathbf{w}$ (iii) $\mathbf{v}\cdot\mathbf{w}$ (iv) $\mathbf{u}\cdot\mathbf{u}$ (v) $\mathbf{v}\cdot\mathbf{v}$ (vi) $\mathbf{w}\cdot\mathbf{w}$
(vii) $|\mathbf{u}|$ (viii) $|\mathbf{v}|$ (ix) $|\mathbf{w}|$ (x) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})$ (xi) $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w})$

2. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be as in the previous exercise. Let α be the angle between $\mathbf{u}$ and $\mathbf{v}$, β be the angle between $\mathbf{u}$ and $\mathbf{w}$, and γ the angle between $\mathbf{v}$ and $\mathbf{w}$. Find

(i) $\cos\alpha$ (ii) $\cos\beta$ (iii) $\cos\gamma$

3. Given that

$$\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}\,, \qquad \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}\,, \qquad \mathbf{c} = 3\mathbf{i} + 6\mathbf{j}\,,$$

determine whether the following are true or false:

(i) The angle between **a** and **b** is acute. (ii) The angle between **b** and **c** is acute.
(iii) The vectors **a** and **c** are mutually perpendicular.
(iv) The angle between the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}-\mathbf{c}$ is obtuse.

Tutorial Exercises:

4. Given that $P = (8,4,-1)$, $Q = (6,3,-4)$ and $R = (7,5,-5)$, find

$$\overrightarrow{QP}\,, \qquad |\overrightarrow{QP}|\,, \qquad \overrightarrow{QR}\,, \qquad |\overrightarrow{QR}|\,, \qquad \overrightarrow{QP}\cdot\overrightarrow{QR}\,,$$

and the cosine of $\angle PQR$.

5. Given that $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$, find

(i) $\mathbf{u}\cdot\mathbf{v}$ (ii) $\hat{\mathbf{u}}$ (iii) $\hat{\mathbf{v}}$ (iv) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|}$ (v) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$ (vi) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$
(vii) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\,\mathbf{u}$ (viii) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}$ (ix) $\mathbf{v} - \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\,\mathbf{u}$ (x) $\mathbf{u} - \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}$
(xi) the cosine of the angle between **u** and **v**
(xii) the scalar component of **u** in the direction of **v**
(xiii) the scalar component of **v** in the direction of **u**
(xiv) the vector projection of **u** in the direction of **v**
(xv) the vector projection of **v** in the direction of **u**
(xvi) the vector component of **u** orthogonal to **v**
(xvii) the vector component of **v** orthogonal to **u**

6. Use the dot product to verify that if $\mathbf{v}$ and $\mathbf{w}$ are any vectors and $\mathbf{w}$ is nonzero, then

$$\mathbf{w} \qquad \text{and} \qquad \mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\,\mathbf{w}$$

are mutually perpendicular.

7. Given that $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, find

(i) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$
(ii) the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$
(iii) the scalar component of $\mathbf{v}$ in the direction of $\mathbf{u}$
(iv) the vector projection of $\mathbf{u}$ in the direction of $\mathbf{v}$
(v) the vector projection of $\mathbf{v}$ in the direction of $\mathbf{u}$
(vi) the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$
(vii) the vector component of $\mathbf{v}$ orthogonal to $\mathbf{u}$

8. Verify that if $\mathbf{a}$ and $\mathbf{b}$ are vectors of the same length then

$$\mathbf{a} + \mathbf{b} \qquad \text{and} \qquad \mathbf{a} - \mathbf{b}$$

are mutually perpendicular.

9. (suitable for group discussion) Use vectors to find the following angles in a cube:

(i) between a major diagonal (between opposite vertices) and an edge,
(ii) between a major diagonal and a face diagonal,
(iii) between diagonals on adjacent faces,
(iv) between major diagonals.

10.* Use vectors to show that any angle inscribed in a semicircle is a right angle.

Further Exercises:

11. Resolve the vector $\mathbf{u} = 5\mathbf{i} + \mathbf{j} + 6\mathbf{k}$ into a sum of two vectors, one of which is parallel and the other perpendicular to $\mathbf{v} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$.

12. *(Homework)* Find the (vector) components of the force $15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k}$ newtons in the direction of and orthogonal to

(i) $-\mathbf{i} + \mathbf{j}$ (ii) $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

13. Prove that if **a** and **b** are mutually perpendicular vectors then

$$|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 .$$

Interpret this result in terms of a well-known fact about triangles.

14.* *(Homework)* Verify that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.

15.* Prove that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus (that is, has all sides of equal length).

Answers to Preparatory Exercises:

1. (i) 6 (ii) 5 (iii) 1 (iv) 6 (v) 9 (vi) 10 (vii) $\sqrt{6}$ (viii) 3

(ix) $\sqrt{10}$ (x) 11 (xi) 1

2. (i) $\dfrac{\sqrt{6}}{3}$ (ii) $\dfrac{\sqrt{15}}{6}$ (iii) $\dfrac{\sqrt{10}}{30}$

3. (i) false (ii) true (iii) true (iv) true