Summer School **Excercise Sheet 3 Solutions** 2012

4. We have

$$\overrightarrow{QP} = 2\mathbf{i}+\mathbf{j}+3\mathbf{k}\,, \qquad |\overrightarrow{QP}| = \sqrt{14}\,, \qquad \overrightarrow{QR} = \mathbf{i}+2\mathbf{j}-\mathbf{k}\,,$$

$$|\overrightarrow{QR}| = \sqrt{6}\,, \qquad \overrightarrow{QP}\cdot\overrightarrow{QR} = 2(1)+1(2)+3(-1) = 1\,,$$

so that the cosine of $\angle PQR$ is $\dfrac{1}{\sqrt{6}\sqrt{14}} = \dfrac{1}{2\sqrt{21}}$.

5. (i) -4 (ii) $\frac{1}{\sqrt5}(\mathbf{i}-2\mathbf{j})$ (iii) $\frac{1}{\sqrt5}(-2\mathbf{i}+\mathbf{j})$ (iv) $-\frac{4}{\sqrt5}$ (v) $-\frac{4}{\sqrt5}$ (vi) $-\frac45$
(vii) $-\frac45(\mathbf{i}-2\mathbf{j})$ (viii) $\frac45(2\mathbf{i}-\mathbf{j})$ (ix) $-\frac35(2\mathbf{i}+\mathbf{j})$ (x) $-\frac35(\mathbf{i}+2\mathbf{j})$

(xi) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ equals $-\frac45$.

(xii) the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ equals $-\frac{4}{\sqrt5}$.

(xiii) the scalar component of $\mathbf{v}$ in the direction of $\mathbf{u}$ equals $-\frac{4}{\sqrt5}$.

(xiv) the vector projection of $\mathbf{u}$ in the direction of $\mathbf{v}$ equals $\frac45(2\mathbf{i}-\mathbf{j})$.

(xv) the vector projection of $\mathbf{v}$ in the direction of $\mathbf{u}$ equals $-\frac45(\mathbf{i}-2\mathbf{j})$.

(xvi) the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ equals $-\frac35(\mathbf{i}+2\mathbf{j})$.

(xvii) the vector component of $\mathbf{v}$ orthogonal to $\mathbf{u}$ equals $-\frac35(2\mathbf{i}+\mathbf{j})$.

6. The vectors are perpendicular because

$$\mathbf{w}\cdot\left(\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\,\mathbf{w}\right) = \mathbf{w}\cdot\mathbf{v}-\mathbf{w}\cdot\left(\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right) = \mathbf{w}\cdot\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}(\mathbf{w}\cdot\mathbf{w})$$

$$= \mathbf{w}\cdot\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}|\mathbf{w}|^2 = \mathbf{w}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{w}$$

$$= \mathbf{w}\cdot\mathbf{v}-\mathbf{w}\cdot\mathbf{v} = 0\,.$$

7. (i) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ equals $\frac{2}{3\sqrt{33}}$.

(ii) the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ equals $\frac{2}{\sqrt{33}}$.

(iii) the scalar component of $\mathbf{v}$ in the direction of $\mathbf{u}$ equals $\frac23$.

(iv) the vector projection of $\mathbf{u}$ in the direction of $\mathbf{v}$ equals $\frac{2}{33}(-4\mathbf{i}+4\mathbf{j}-\mathbf{k})$.

(v) the vector projection of $\mathbf{v}$ in the direction of $\mathbf{u}$ equals $\frac29(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$.

(vi) the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ equals $\frac{1}{33}(41\mathbf{i}+58\mathbf{j}+68\mathbf{k})$.

(vii) the vector component of $\mathbf{v}$ orthogonal to $\mathbf{u}$ equals $-\frac19(38\mathbf{i}-32\mathbf{j}+13\mathbf{k})$.

8. Observe that

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b} \\ &= |\mathbf{a}|^2-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}-|\mathbf{b}|^2 \\ &= |\mathbf{a}|^2-|\mathbf{a}|^2 \;=\; 0\,,\end{aligned}$$

so that $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are mutually perpendicular.

9. (i) The cosine of the angle between a major diagonal and an edge is

$$\frac{\left|\mathbf{i}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\right|}{|\mathbf{i}|\,|\mathbf{i}+\mathbf{j}+\mathbf{k}|} \;=\; \frac{1}{\sqrt{3}}\,,$$

yielding an angle of approximately 54 degrees.

(ii) The cosine of the angle between a major diagonal and a face diagonal is

$$\frac{\left|(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\right|}{|\mathbf{i}+\mathbf{j}|\,|\mathbf{i}+\mathbf{j}+\mathbf{k}|} \;=\; \frac{2}{\sqrt{6}}\,,$$

yielding an angle of approximately 35 degrees.

(iii) The cosine of the angle between diagonals on adjacent faces is

$$\frac{\left|(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{k})\right|}{|\mathbf{i}+\mathbf{j}|\,|\mathbf{i}+\mathbf{k}|} \;=\; \frac{1}{2}\,,$$

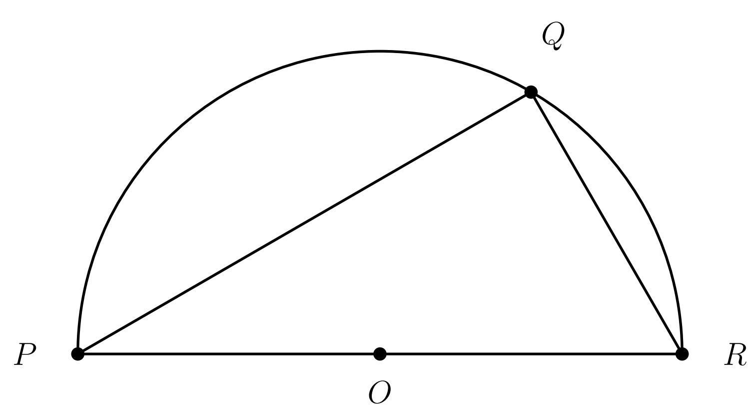
yielding an angle of exactly 60 degrees.

(iv) The cosine of the angle between major diagonals is

$$\frac{\left|(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+\mathbf{k})\right|}{|\mathbf{i}+\mathbf{j}+\mathbf{k}|\,|\mathbf{i}-\mathbf{j}+\mathbf{k}|} \;=\; \frac{1}{3}\,,$$

yielding an angle of approximately 71 degrees.

10.* Consider a semicircle and points O, P, Q, R as shown:



To show the angle at Q is a right-angle, it suffices to show $\overrightarrow{QP} \cdot \overrightarrow{QR} = 0$. But this follows from Exercise 8, since

$$\overrightarrow{QP} = \overrightarrow{QO} + \overrightarrow{OP} \qquad \text{and} \qquad \overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR} = \overrightarrow{QO} - \overrightarrow{OP}$$

and the fact that $|\overrightarrow{QO}| = |\overrightarrow{OP}|$, the radius of the circle.

11. The projection of $\mathbf{u}$ in the direction of $\mathbf{v}$ is

$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{15 - 6 + 12}{9 + 36 + 4}\mathbf{v} = \frac{3}{7}\big(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}\big) .$$

The component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ is

$$\mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \mathbf{u} - \frac{3}{7}\big(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}\big) = \frac{1}{7}\big(26\mathbf{i} + 25\mathbf{j} + 36\mathbf{k}\big) .$$

Thus, the decomposition of $\mathbf{u}$ as a sum of vectors, the first parallel to $\mathbf{v}$ and the second perpendicular to $\mathbf{v}$, is

$$\mathbf{u} = \frac{3}{7}\big(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}\big) + \frac{1}{7}\big(26\mathbf{i} + 25\mathbf{j} + 36\mathbf{k}\big) .$$

12. (i) The component of the force in the direction of $-\mathbf{i} + \mathbf{j}$ is

$$\frac{\big(15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k}\big) \cdot (-\mathbf{i} + \mathbf{j})}{\big| - \mathbf{i} + \mathbf{j}\big|^2}(-\mathbf{i} + \mathbf{j}) = \frac{5}{2}(-\mathbf{i} + \mathbf{j}) \text{ newtons} ,$$

and the component orthogonal to $-\mathbf{i} + \mathbf{j}$ is

$$15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k} - \frac{5}{2}(-\mathbf{i} + \mathbf{j}) = \frac{1}{2}\big(35\mathbf{i} + 35\mathbf{j} + 12\mathbf{k}\big) \text{ newtons} .$$

(ii) The component of the force in the direction of $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is

$$\frac{\big(15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k}\big) \cdot \big(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\big)}{\big|2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\big|^2}\big(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\big) = -\frac{12}{7}\big(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\big) \text{ newtons} ,$$

and the component orthogonal to $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is

$$15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k} + \frac{12}{7}\big(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\big) = \frac{1}{7}\big(129\mathbf{i} + 104\mathbf{j} + 54\mathbf{k}\big) \text{ newtons} .$$

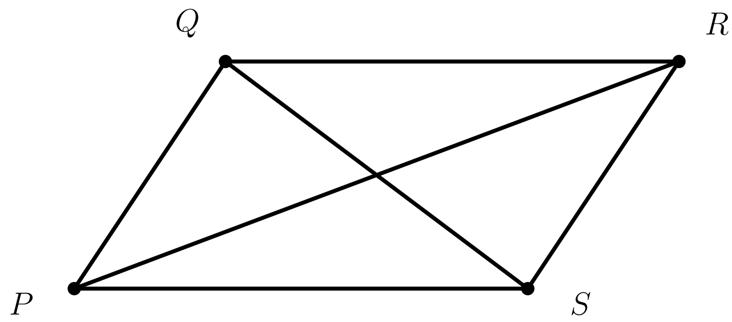
13. If $\mathbf{a}$ and $\mathbf{b}$ are mutually perpendicular then $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = 0$, so that

$$|\mathbf{a} + \mathbf{b}|^2 = \big(\mathbf{a} + \mathbf{b}\big) \cdot \big(\mathbf{a} + \mathbf{b}\big) = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}$$

$$= |\mathbf{a}|^2 + 0 + 0 + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 .$$

This is just the usual Theorem of Pythagoras where $\mathbf{a}$ and $\mathbf{b}$ label directed edges of a right-angled triangle.

14.* Consider the following parallelogram, and put $\mathbf{v} = \overrightarrow{PQ}$ and $\mathbf{w} = \overrightarrow{QR}$:



Then the sum of the squares of the lengths of the diagonals is

$$\begin{aligned}
|\overrightarrow{PR}|^2 + |\overrightarrow{QS}|^2 &= \overrightarrow{PR} \cdot \overrightarrow{PR} + \overrightarrow{QS} \cdot \overrightarrow{QS} \\
&= (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) + (\mathbf{w} - \mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}) \\
&= \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} - 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} \\
&= 2\mathbf{v} \cdot \mathbf{v} + 2\mathbf{w} \cdot \mathbf{w} = 2\big(|\mathbf{v}|^2 + |\mathbf{w}|^2\big) ,
\end{aligned}$$

which is the sum of the squares of the lengths of the sides.

15.* The diagonals of the parallelogram of the previous exercise are perpendicular if and only if

$$\overrightarrow{QS} \cdot \overrightarrow{PR} = 0 ,$$

that is,

$$(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} = |\mathbf{v}|^2 - |\mathbf{w}|^2 = 0 ,$$

that is,

$$|\mathbf{v}|^2 = |\mathbf{w}|^2 ,$$

that is,

$$|\mathbf{v}| = |\mathbf{w}| ,$$

that is, the parallelogram is a rhombus.