THE UNIVERSITY OF SYDNEY

MATH1002 LINEAR ALGEBRA

Summer School **Exercise Sheet 4** 2012

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Algebraic definition of cross product: If $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$ then

$$\mathbf{v} \times \mathbf{w} \;=\; (v_2w_3 - v_3w_2)\,\mathbf{i} + (v_3w_1 - v_1w_3)\,\mathbf{j} + (v_1w_2 - v_2w_1)\,\mathbf{k}\,.$$

which can be evaluated by

(a) using the "up-and-down-diagonal" method;

(b) using the "expanding brackets" method and the facts that

$$\mathbf{i} \times \mathbf{j} \;=\; \mathbf{k} \;=\; -\big(\mathbf{j} \times \mathbf{i}\big)\,, \quad \mathbf{j} \times \mathbf{k} \;=\; \mathbf{i} \;=\; -\big(\mathbf{k} \times \mathbf{j}\big)\,, \quad \mathbf{k} \times \mathbf{i} \;=\; \mathbf{j} \;=\; -\big(\mathbf{i} \times \mathbf{k}\big)\,,$$

$$\mathbf{i} \times \mathbf{i} \;=\; \mathbf{j} \times \mathbf{j} \;=\; \mathbf{k} \times \mathbf{k} \;=\; 0\,;$$

(c) evaluating a 3×3 deteminant (see Week 11): $\mathbf{v} \times \mathbf{w} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\,.$

(ii) The cross product $\mathbf{v} \times \mathbf{w}$ is always perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ so that

$$\big(\mathbf{v} \times \mathbf{w}\big) \cdot \mathbf{v} \;=\; \big(\mathbf{v} \times \mathbf{w}\big) \cdot \mathbf{w} \;=\; 0\,.$$

(iii) Anti-commutativity of cross product: $\mathbf{v} \times \mathbf{w} \;=\; -\big(\mathbf{w} \times \mathbf{v}\big)\,.$

(iv) Distributivity of cross over plus: $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} \;=\; \mathbf{u} \times \mathbf{w} \,+\, \mathbf{v} \times \mathbf{w}\,.$

(v) If $\mathbf{v}$ and $\mathbf{w}$ are vectors and λ is a scalar then

$$(\lambda\mathbf{v}) \times \mathbf{w} = \lambda(\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (\lambda\mathbf{w}) \qquad \text{and} \qquad \mathbf{v} \times \mathbf{v} = \mathbf{0}\,.$$

(vi) The area of the parallelogram inscribed by $\mathbf{v}$ and $\mathbf{w}$ is $|\mathbf{v} \times \mathbf{w}|\,.$

(vii) The area of the triangle inscribed by $\mathbf{v}$ and $\mathbf{w}$ is $\dfrac{|\mathbf{v} \times \mathbf{w}|}{2}\,.$

(viii) Geometric formula for cross product: if θ is the angle between vectors $\mathbf{v}$ and $\mathbf{w}$ chosen so that $0 \le \theta \le \pi$ then

$$\mathbf{v} \times \mathbf{w} \;=\; |\mathbf{v}||\mathbf{w}|\sin\theta\;\mathbf{u}\,,$$

where $\mathbf{u}$ is the unit vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ such that the triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is right-handed. In particular

$$|\mathbf{v} \times \mathbf{w}| \;=\; |\mathbf{v}||\mathbf{w}|\sin\theta\,.$$

Preparatory Exercises:

1. Write down

(i) $\mathbf{i}\times\mathbf{j}$ (ii) $2\mathbf{i}\times 3\mathbf{j}$ (iii) $\mathbf{i}\times(-4\mathbf{j})$ (iv) $\mathbf{j}\times\mathbf{i}$ (v) $\mathbf{j}\times(-4\mathbf{i})$
(vi) $\mathbf{j}\times\mathbf{k}$ (vii) $\mathbf{k}\times\mathbf{k}$ (viii) $\mathbf{k}\times(-\mathbf{k})$ (ix) $(-\mathbf{k})\times\mathbf{i}$ (x) $(-\mathbf{k})\times(-\mathbf{j})$
(xi) $\mathbf{k}\times(\mathbf{i}+\mathbf{k})$ (xii) $(3\mathbf{j}-\mathbf{k})\times 2\mathbf{j}$ (xiii) $(\mathbf{j}-\mathbf{k})\times(\mathbf{k}+\mathbf{j})$

2. Evaluate

(i) $\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$ (ii) $(\mathbf{i}+\mathbf{j}+\mathbf{k})\times(2\mathbf{i}+\mathbf{k})$
(iii) $(2\mathbf{i}+4\mathbf{j}-3\mathbf{k})\times(2\mathbf{i}-3\mathbf{j}+4\mathbf{k})$ (iv) $(\mathbf{i}-\mathbf{j}+3\mathbf{k})\times(3\mathbf{i}+\mathbf{j}-\mathbf{k})$

3. Given that

$$\mathbf{a} = 2\mathbf{i}-\mathbf{j}+2\mathbf{k}\,, \qquad \mathbf{b} = \mathbf{i}+\mathbf{j}-\mathbf{k}\,,$$

find

(i) $|\mathbf{a}|$ (ii) $|\mathbf{b}|$ (iii) $\mathbf{a}\times\mathbf{b}$ (iv) $|\mathbf{a}\times\mathbf{b}|$
(v) the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$.

Tutorial Exercises:

4. Find two unit vectors perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ where

$$\mathbf{v}=\mathbf{i}+2\mathbf{j}-7\mathbf{k} \qquad \text{and} \qquad \mathbf{w}=5\mathbf{i}+\mathbf{j}+\mathbf{k}\,.$$

5. Given that $P=(8,4,-1)$, $Q=(6,3,-4)$ and $R=(7,5,-5)$, find

$$\overrightarrow{QP}\times\overrightarrow{QR}$$

and the area of the triangle $\triangle PQR$.

6. Given that $\mathbf{a}=2\mathbf{i}-\mathbf{j}$, $\mathbf{b}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{c}=-2\mathbf{i}+\mathbf{k}$ find

(i) $\mathbf{a}\times\mathbf{b}$ (ii) $\mathbf{a}\times\mathbf{c}$ (iii) $\mathbf{b}\times\mathbf{c}$ (iv) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ (v) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$
(vi) $\mathbf{a}\times(\mathbf{a}\times\mathbf{c})$ (vii) $\mathbf{a}\times(\mathbf{a}+\mathbf{c})$ (viii) $(\mathbf{a}\times\mathbf{a})\times\mathbf{c}$ (ix) $\mathbf{a}\times(\mathbf{b}-2\mathbf{c})$
(x) the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$
(xi) the area of the parallelogram inscribed by $\mathbf{a}$ and $\mathbf{c}$
(xii) the area of the triangle inscribed by $\mathbf{b}$ and $\mathbf{c}$

7. Given that $\mathbf{v}$ and $\mathbf{w}$ are vectors such that $\mathbf{v}\times\mathbf{w}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ find

(i) $\mathbf{w}\times\mathbf{v}$ (ii) $(\mathbf{v}+3\mathbf{w})\times(2\mathbf{w}-\mathbf{v})$

8. Calculate $|\mathbf{a} \times \mathbf{b}|$ given that $|\mathbf{a}| = 7$, $|\mathbf{b}| = 4$ and $\mathbf{a} \cdot \mathbf{b} = -21$.

9. (suitable for group discussion) A tetrahedron has four faces. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ be vectors perpendicular to the faces, pointing outwards, of length equal to the respective areas of the faces. Verify that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 \;=\; \mathbf{0}\,.$$

10.* Verify that if **a** and **b** are geometric vectors then the following "correction" to the Cauchy-Schwarz Inequality holds:

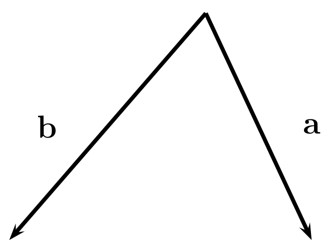
$$\sqrt{|\mathbf{a} \cdot \mathbf{b}|^2 + |\mathbf{a} \times \mathbf{b}|^2} \;=\; |\mathbf{a}|\,|\mathbf{b}|\,.$$

Further Exercises:

11. Find the areas of the triangles having vertices

(i) $(0,0,0)\,,\ (2,2,-1)\,,\ (3,-4,2)$ (ii) $(3,-1,2)\,,\ (1,-1,-3)\,,\ (4,-3,1)$

12. Let **a** and **b** be the following vectors in the page:



True or false:

(i) $\mathbf{a} \times \mathbf{b}$ points upwards, away from the page, towards the ceiling

(ii) $\mathbf{b} \times (\mathbf{a} - \mathbf{b})$ points downwards, away from the page, towards the floor

(iii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{a})$ is perpendicular to **a** but not to **b**

(iv) $\mathbf{b} \times (\mathbf{b} \times \mathbf{a})$ is the zero vector

13. Use the cross product to find

(i) a unit vector perpendicular to both $-\mathbf{i} + 2\mathbf{j}$ and $\mathbf{j} + 3\mathbf{k}$,

(ii)* a unit vector which points in a direction which is perpendicular to the triangle with vertices

$$A(0,0,-1)\,,\quad B(1,-2,-1)\,,\quad C(1,-3,-4)$$

such that looking backwards along the vector (from tip to tail) towards the triangle, the vertices A, B, C rotate anticlockwise (in that order).

14.* Verify that, for any geometric vectors **a**, **b**, **c**,

$$\mathbf{a}\cdot\big(\mathbf{b}\times\mathbf{c}\big) \;=\; \big(\mathbf{a}\times\mathbf{b}\big)\cdot\mathbf{c}\,.$$

15.* Verify that, for any geometric vectors **a**, **b**, **c**,

$$\big(\mathbf{a}\times\mathbf{b}\big)\times\mathbf{c} \;=\; \big(\mathbf{a}\cdot\mathbf{c}\big)\mathbf{b}-\big(\mathbf{b}\cdot\mathbf{c}\big)\mathbf{a}\,.$$

Use anti-commutativity to deduce that

$$\mathbf{a}\times\big(\mathbf{b}\times\mathbf{c}\big) \;=\; \big(\mathbf{a}\cdot\mathbf{c}\big)\mathbf{b}-\big(\mathbf{a}\cdot\mathbf{b}\big)\mathbf{c}\,.$$

Short Answers to Selected Exercises:

1. (i) $\mathbf{k}$ (ii) $6\mathbf{k}$ (iii) $-4\mathbf{k}$ (iv) $-\mathbf{k}$ (v) $4\mathbf{k}$ (vi) $\mathbf{i}$ (vii) $\mathbf{0}$ (viii) $\mathbf{0}$

(ix) $-\mathbf{j}$ (x) $-\mathbf{i}$ (xi) $\mathbf{j}$ (xii) $2\mathbf{i}$ (xiii) $2\mathbf{i}$

2. (i) $\mathbf{k}-\mathbf{j}$ (ii) $\mathbf{i}+\mathbf{j}-2\mathbf{k}$ (iii) $7\mathbf{i}-14\mathbf{j}-14\mathbf{k}$ (iv) $-2\mathbf{i}+10\mathbf{j}+4\mathbf{k}$

3. (i) 3 (ii) $\sqrt{3}$ (iii) $-\mathbf{i}+4\mathbf{j}+3\mathbf{k}$ (iv) $\sqrt{26}$ (v) $\dfrac{\sqrt{78}}{9}$

4. $\pm\dfrac{\sqrt{2}}{6}\big(\mathbf{i}-4\mathbf{j}-\mathbf{k}\big)$ **5.** $-7\mathbf{i}+5\mathbf{j}+3\mathbf{k}\,,\ \dfrac{\sqrt{83}}{2}$

6. (i) $-\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ (ii) $-\mathbf{i}-2\mathbf{j}-2\mathbf{k}$ (iii) $\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ (iv) $-2\mathbf{i}-4\mathbf{j}-5\mathbf{k}$

(v) $-2\mathbf{i}-5\mathbf{j}-4\mathbf{k}$ (vi) $2\mathbf{i}+4\mathbf{j}-5\mathbf{k}$ (vii) $-\mathbf{i}-2\mathbf{j}-2\mathbf{k}$ (viii) $\mathbf{0}$ (ix) $\mathbf{i}+2\mathbf{j}+7\mathbf{k}$

(x) $\sqrt{14}/\sqrt{15}$ (xi) 3 (xii) $\sqrt{14}/2$

7. (i) $-2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ (ii) $10\mathbf{i}-5\mathbf{j}+15\mathbf{k}$ **8.** $7\sqrt{7}$

11. (i) $7\sqrt{5}/2$ (ii) $\sqrt{165}/2$ **12.** (i) False (ii) False (iii) True (iv) False

13. (i) $\pm\dfrac{1}{\sqrt{46}}\big(6\mathbf{i}+3\mathbf{j}-\mathbf{k}\big)\,.$