Summer School **Excercise Sheet 4 Solutions** 2012

4. Observe that

$$\mathbf{v} \times \mathbf{w} \;=\; \big(\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}\big) \times \big(5\mathbf{i} + \mathbf{j} + \mathbf{k}\big) \;=\; 9\mathbf{i} - 36\mathbf{j} - 9\mathbf{k} \;=\; 9\big(\mathbf{i} - 4\mathbf{j} - \mathbf{k}\big)\,,$$

which has length $9\sqrt{1+16+1} \;=\; 9\sqrt{18} \;=\; 27\sqrt{2}$. Hence two unit vectors perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ are

$$\pm\frac{\sqrt{2}}{6}\big(\mathbf{i} - 4\mathbf{j} - \mathbf{k}\big)\,.$$

5. We have

$$\overrightarrow{QP} \times \overrightarrow{QR} \;=\; \big(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}\big) \times \big(\mathbf{i} + 2\mathbf{j} - \mathbf{k}\big) \;=\; -7\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}\,.$$

Hence the area of the triangle $\triangle PQR$ is

$$\frac{\big|7\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}\big|}{2} \;=\; \frac{\sqrt{83}}{2}\,.$$

6. (i) $\mathbf{a} \times \mathbf{b} \;=\; -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}\,.$ (ii) $\mathbf{a} \times \mathbf{c} \;=\; -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}\,.$

(iii) $\mathbf{b} \times \mathbf{c} \;=\; \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}\,.$ (iv) $\mathbf{a} \times \big(\mathbf{b} \times \mathbf{c}\big) \;=\; -2\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}\,.$

(v) $\big(\mathbf{a} \times \mathbf{b}\big) \times \mathbf{c} \;=\; -2\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}\,.$ (vi) $\mathbf{a} \times \big(\mathbf{a} \times \mathbf{c}\big) \;=\; 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}\,.$

(vii) $\mathbf{a} \times \big(\mathbf{a} + \mathbf{c}\big) \;=\; \mathbf{a} \times \mathbf{c} \;=\; -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ (viii) $\big(\mathbf{a} \times \mathbf{a}\big) \times \mathbf{c} \;=\; \mathbf{0}\,.$

(ix) $\mathbf{a} \times \big(\mathbf{b} - 2\mathbf{c}\big) \;=\; \mathbf{a} \times \mathbf{b} - 2(\mathbf{a} \times \mathbf{c}) \;=\; \mathbf{i} + 2\mathbf{j} + 7\mathbf{k}\,.$

(x) the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$ equals $\dfrac{\big|\mathbf{a} \times \mathbf{b}\big|}{|\mathbf{a}|\,|\mathbf{b}|} \;=\; \dfrac{\sqrt{14}}{\sqrt{15}}\,.$

(xi) the area of the parallelogram inscribed by $\mathbf{a}$ and $\mathbf{c}$ equals $\big|\mathbf{a} \times \mathbf{c}\big| \;=\; 3\,.$

(xii) the area of the triangle inscribed by $\mathbf{b}$ and $\mathbf{c}$ equals $\dfrac{\big|\mathbf{b} \times \mathbf{c}\big|}{2} \;=\; \dfrac{\sqrt{14}}{2}\,.$

7. (i) $\mathbf{w} \times \mathbf{v} \;=\; -\big(\mathbf{v} \times \mathbf{w}\big) \;=\; -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}\,.$

(ii) $\big(\mathbf{v} + 3\mathbf{w}\big) \times \big(2\mathbf{w} - \mathbf{v}\big) \;=\; 2\big(\mathbf{v} \times \mathbf{w}\big) - 3\big(\mathbf{w} \times \mathbf{v}\big) \;=\; 5\big(\mathbf{v} \times \mathbf{w}\big) \;=\; 10\mathbf{i} - 5\mathbf{j} + 15\mathbf{k}\,.$

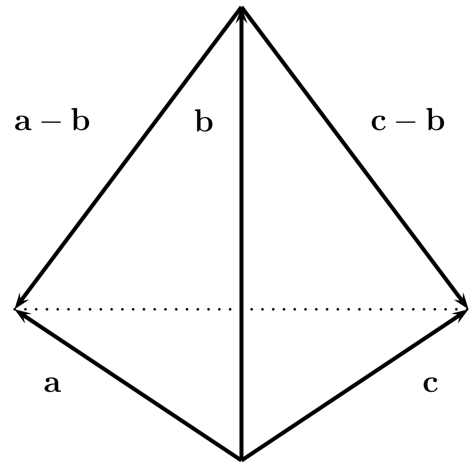
8. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$ measured between 0 and 180 degrees. Then

$$\cos\theta \;=\; \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \;=\; \frac{-21}{28} \;=\; -\frac{3}{4}\,,$$

so $\sin\theta \;=\; \sqrt{1 - \dfrac{9}{16}} \;=\; \dfrac{\sqrt{7}}{4}$. Hence

$$|\mathbf{a} \times \mathbf{b}| \;=\; |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta \;=\; 7\sqrt{7}\,.$$

9. Consider a tetrahedron with directed edges labelled as follows



Using the Right-Hand Rule for successive faces, we may take

$$\mathbf{v}_1 = \frac{1}{2}(\mathbf{b}\times\mathbf{a}),\quad \mathbf{v}_2 = \frac{1}{2}(\mathbf{c}\times\mathbf{b}),\quad \mathbf{v}_3 = \frac{1}{2}(\mathbf{a}\times\mathbf{c}),\quad \mathbf{v}_4 = \frac{1}{2}\Big[(\mathbf{c}-\mathbf{b})\times(\mathbf{a}-\mathbf{b})\Big].$$

Hence

$$\begin{aligned} 2(\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4) &= \mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+(\mathbf{c}-\mathbf{b})\times(\mathbf{a}-\mathbf{b}) \\ &= \mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}-\mathbf{c}\times\mathbf{b}-\mathbf{b}\times\mathbf{a} \\ &= \mathbf{a}\times\mathbf{c}-\mathbf{a}\times\mathbf{c} = \mathbf{0}. \end{aligned}$$

This shows $\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4 = \mathbf{0}$.

10.* Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Using the geometric formulae for the dot and cross product, we get

$$\begin{aligned} \sqrt{|\mathbf{a}\cdot\mathbf{b}|^2+|\mathbf{a}\times\mathbf{b}|^2} &= \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta+|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta} \\ &= \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2(\cos^2+\sin^2\theta)} = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2} = |\mathbf{a}|\,|\mathbf{b}|. \end{aligned}$$

11. (i) Area equals $\dfrac{\big|(2\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})\big|}{2} = \dfrac{\big|-7\mathbf{j}-14\mathbf{k}\big|}{2} = \dfrac{7\sqrt{5}}{2}$.

(ii) Area equals $\dfrac{\big|(-2\mathbf{i}-5\mathbf{k})\times(\mathbf{i}-2\mathbf{j}-\mathbf{k})\big|}{2} = \dfrac{\big|-10\mathbf{i}-7\mathbf{j}+4\mathbf{k}\big|}{2} = \dfrac{\sqrt{165}}{2}$.

12. (i) False (ii) False (iii) True (iv) False

13. (i) Observe that

$$\big(-\mathbf{i}+2\mathbf{j}\big)\times\big(\mathbf{j}+3\mathbf{k}\big) \;=\; 6\mathbf{i}+3\mathbf{j}-\mathbf{k}\,,$$

so a unit vector perpendicular to both $-\mathbf{i}+2\mathbf{j}$ and $\mathbf{j}+3\mathbf{k}$ is

$$\pm\frac{1}{\sqrt{46}}\big(6\mathbf{i}+3\mathbf{j}-\mathbf{k}\big)\,.$$

(ii)* We want a unit vector pointing in the direction of

$$\overrightarrow{BC}\times\overrightarrow{BA} \;=\; \big(-\mathbf{j}-3\mathbf{k}\big)\times\big(-\mathbf{i}+2\mathbf{j}\big) \;=\; \big(-\mathbf{i}+2\mathbf{j}\big)\times\big(\mathbf{j}+3\mathbf{k}\big)\,,$$

which is $\dfrac{1}{\sqrt{46}}\big(6\mathbf{i}+3\mathbf{j}-\mathbf{k}\big)$.

14.* Put $\mathbf{a}=a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k}$, $\mathbf{b}=b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k}$ and $\mathbf{c}=c_1\,\mathbf{i}+c_2\,\mathbf{j}+c_3\,\mathbf{k}$. Then, by the algebraic formulae for dot and cross products,

$$\begin{aligned}
\mathbf{a}\cdot\big(\mathbf{b}\times\mathbf{c}\big) &= a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)\\
&= a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1\\
&= (a_2b_3-a_3b_2)c_1+(a_3b_1-a_1b_3)c_2+(a_1b_2-a_2b_1)c_3\\
&= \big(\mathbf{a}\times\mathbf{b}\big)\cdot\mathbf{c}\,.
\end{aligned}$$

15.* Put $\mathbf{a}=a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k}$, $\mathbf{b}=b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k}$ and $\mathbf{c}=c_1\,\mathbf{i}+c_2\,\mathbf{j}+c_3\,\mathbf{k}$. Then, by the algebraic formulae for dot and cross products,

$$\begin{aligned}
\big(\mathbf{a}\times\mathbf{b}\big)\times\mathbf{c} &= \big[(a_2b_3-a_3b_2)\,\mathbf{i}+(a_3b_1-a_1b_3)\,\mathbf{j}+(a_1b_2-a_2b_1)\,\mathbf{k}\big]\times\mathbf{c}\\
&= \big[(a_3b_1-a_1b_3)c_3-(a_1b_2-a_2b_1)c_2\big]\mathbf{i}+\big[(a_1b_2-a_2b_1)c_1-(a_2b_3-a_3b_2)c_3\big]\mathbf{j}\\
&\qquad+\big[(a_2b_3-a_3b_2)c_2-(a_3b_1-a_1b_3)c_1\big]\mathbf{k}\\
&= (a_2c_2+a_3c_3)b_1\,\mathbf{i}+(a_1c_1+a_3c_3)b_2\,\mathbf{j}+(a_1c_1+a_2c_2)b_3\,\mathbf{k}\\
&\qquad-(b_2c_2+b_3c_3)a_1\,\mathbf{i}-(b_1c_1+b_3c_3)a_2\,\mathbf{j}-(b_1c_1+b_2c_2)a_3\,\mathbf{k}\\
&= (a_1c_1+a_2c_2+a_3c_3)b_1\,\mathbf{i}+(a_1c_1+a_2c_2+a_3c_3)b_2\,\mathbf{j}+(a_1c_1+a_2c_2+a_3c_3)b_3\,\mathbf{k}\\
&\qquad-(b_1c_1+b_2c_2+b_3c_3)a_1\,\mathbf{i}-(b_1c_1+b_2c_2+b_3c_3)a_2\,\mathbf{j}-(b_1c_1+b_2c_2+b_3c_3)a_3\,\mathbf{k}\\
&= \big(\mathbf{a}\cdot\mathbf{c}\big)\mathbf{b}-\big(\mathbf{b}\cdot\mathbf{c}\big)\mathbf{a}\,.
\end{aligned}$$

By anti-commutativity of the cross product, commutativity of the dot product, and what we have just proved,

$$\mathbf{a}\times\big(\mathbf{b}\times\mathbf{c}\big) \;=\; -\big[\big(\mathbf{b}\times\mathbf{c}\big)\times\mathbf{a}\big] \;=\; -\big[\big(\mathbf{b}\cdot\mathbf{a}\big)\mathbf{c}-\big(\mathbf{c}\cdot\mathbf{a}\big)\mathbf{b}\big] \;=\; \big(\mathbf{a}\cdot\mathbf{c}\big)\mathbf{b}-\big(\mathbf{a}\cdot\mathbf{b}\big)\mathbf{c}\,.$$