THE UNIVERSITY OF SYDNEY

MATH1002 LINEAR ALGEBRA

Summer School	**Exercise Sheet 5**	2012

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) A line in space is determined by two points, or by one point and a direction.

(ii) A plane in space is determined either by three non-collinear points, or by one point and a perpendicular (normal) direction.

(iii) If the vector $\mathbf{v}$ points in the direction of a line $\mathcal{L}$ containing the point P_0, then the *parametric vector equation* of $\mathcal{L}$ is

$$\mathbf{r} - \mathbf{r}_0 = t\mathbf{v} \qquad \text{or equivalently} \qquad \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}$ is the position vector of a typical point on $\mathcal{L}$, $\mathbf{r}_0$ is the position vector of P_0 and t is a parameter which varies over all real numbers.

(iv) If the vector $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ points in the direction of a line $\mathcal{L}$ containing the point $P_0(x_0, y_0, z_0)$, then the *parametric scalar equations* of $\mathcal{L}$ are

$$\left.\begin{array}{rcl} x &=& x_0 + ta \\ y &=& y_0 + tb \\ z &=& z_0 + tc \end{array}\right\} \; t \in \mathbb{R}$$

and the *Cartesian equations* are (in the case that a, b, c are all nonzero):

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \,.$$

(v) If the vector $\mathbf{n}$ is normal to a plane $\mathcal{P}$ containing the point P_0, then the *vector equation* of $\mathcal{P}$ is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0 \qquad \text{or equivalently} \qquad \mathbf{r} \cdot \mathbf{n} = \mathbf{r}_0 \cdot \mathbf{n}$$

where $\mathbf{r}$ is the position vector of a typical point and $\mathbf{r}_0$ is the position vector of P_0.

(vi) If the vector $\mathbf{n} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is normal to the plane $\mathcal{P}$ containing the point $P_0(x_0, y_0, z_0)$, then the *Cartesian equation* of $\mathcal{P}$ is

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

(vii) If P_1, P_2, P_3 are non-collinear points on a plane, then a normal vector to the plane is

$$\mathbf{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \,.$$

Preparatory Exercises:

1. Find parametric vector, parametric scalar and Cartesian equations of the line passing through the point $(2,3,5)$ in the direction of $\mathbf{i}+3\mathbf{j}-\mathbf{k}$.

2. Find vector and Cartesian equations of the plane containing the point $(2,3,5)$ with normal vector $\mathbf{i}+3\mathbf{j}-\mathbf{k}$.

3. Let $P=(1,2,3)$, $Q=(-1,-2,-3)$ and $R=(4,-4,4)$.

(i) Express $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in Cartesian form.

(ii) Find the cross product $\overrightarrow{PQ}\times\overrightarrow{PR}$.

(iii) Find the Cartesian equation of the plane containing P, Q, R.

Tutorial Exercises:

4. Find parametric vector, parametric scalar and Cartesian equations of the line passing through P in the direction of $\mathbf{v}$ in each of the following cases:

(i) $P=(1,0,-1)\,,\quad \mathbf{v}=2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ (ii) $P=(-6,5,2)\,,\quad \mathbf{v}=2\mathbf{i}-5\mathbf{k}$

(iii) $P=(0,1,-1)\,,\quad \mathbf{v}=\mathbf{i}+2\mathbf{j}$ (iv) $P=(2,3,-3)\,,\quad \mathbf{v}=-\mathbf{i}$

5. Find parametric vector, parametric scalar and Cartesian equations of the line passing through P and Q in each of the following cases:

(i) $P=(-4,3,5)\,,\quad Q=(-2,4,-1)$ (ii) $P=(0,5,0)\,,\quad Q=(5,0,-5)$

6. Find vector and Cartesian equations of the plane containing P having normal vector $\mathbf{n}$ in each of the following cases:

(i) $P=(4,-1,0)\,,\quad \mathbf{n}=3\mathbf{i}+\mathbf{j}-4\mathbf{k}$ (ii) $P=(7,5,-3)\,,\quad \mathbf{n}=2\mathbf{i}+\mathbf{k}$

(iii) $P=(0,0,-9)\,,\quad \mathbf{n}=\mathbf{i}+\mathbf{j}-\mathbf{k}$ (iv) $P=(-6,5,6)\,,\quad \mathbf{n}=\mathbf{j}$

7. Find a Cartesian equation for the plane containing

$$P=(6,7,-2)\,,\qquad Q=(0,-8,11)\,,\qquad R=(14,-3,9)\,.$$

8. *(Homework)* The planes $x+y+z=2$ and $x-y+3z=0$ intersect in a line $\mathcal{L}$.

(i) Find a point on $\mathcal{L}$.

(ii) Use cross products to find a vector pointing in the direction of $\mathcal{L}$.

(iii) Write down parametric scalar and Cartesian equations for $\mathcal{L}$.

9. (suitable for group discussion) Two lines in space are *skew* if they are not parallel and do not intersect. The following lines are not parallel. Show that they are not skew, by finding their point of intersection:

$$\mathcal{L}_1 : \quad \mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \ , \qquad \mathcal{L}_2 : \quad \mathbf{r} = 6\mathbf{i} - 6\mathbf{j} + \mathbf{k} + s(-7\mathbf{i} + 5\mathbf{j} - 6\mathbf{k})$$

10.* Find the distance from $P(3, 0, -1)$ to the plane $\mathcal{P}$ described by the equation

$$4x + 2y - z = 6 \ .$$

Find the closest point to P which lies on $\mathcal{P}$.

Further Exercises:

11. For each of (i)–(vii), find two matching descriptions from (a)–(n).

(i) line containing $(0, 0, 0)$ in the direction of $\mathbf{i} + \mathbf{j} + \mathbf{k}$
(ii) line containing $(-1, 2, -1)$ in the direction of $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
(iii) line containing $(-1, 2, -1)$ and $(0, 0, -2)$
(iv) plane containing $(0, 0, 0)$ with normal vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$
(v) plane containing $(-1, 2, -1)$ with normal vector $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
(vi) plane containing $(-1, 2, -1)$, $(0, 0, -2)$ and $(1, 3, 3)$
(vii) plane containing $(-1, 2, -1)$, $(0, 0, -2)$ and $(1, 3, 2)$

(a) $x + y + z = 0$ (b) $x = y = z$ (c) $x + y - z = 2$

(d) $x + 1 = \dfrac{y - 2}{-2} = \dfrac{z + 1}{-2}$ (e) $7x + 6y - 5z = 10$

(f) $x + 1 = \dfrac{y - 2}{-2} = \dfrac{z + 1}{-1}$ (g) $x - 2y - 2z = -3$

(h) $(\mathbf{r} + 2\mathbf{k}) \cdot (7\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}) = 0$ (i) $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 5\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$

(j) $(\mathbf{r} + 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0$ (k) $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0$

(l) $(\mathbf{r} + 3\mathbf{i}) \cdot (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = 0$ (m) $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$

(n) $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} - t(\mathbf{i} + \mathbf{j} + \mathbf{k})$

12. Find Cartesian equations of the line passing through $(1, 0, -2)$ and perpendicular to the plane $3x - 4y + z = 6$.

13. Verify that the line

$$\frac{x - 3}{2} = \frac{y - 4}{3} = \frac{z - 5}{4}$$

is parallel to the plane $4x + 4y - 5z = 14$.

14. Find the Cartesian equation of the plane containing $(1, 1, 1)$ and the line

$$\frac{x-4}{-2} = y+3 = \frac{z-1}{3}.$$

15.* *(Homework)* Find the distance from $P(2, 1, 1)$ to the line $\mathcal{L}$ given by the equations

$$x-1 = \frac{y-1}{3} = \frac{z+4}{-1}.$$

Find the closest point to P lying on $\mathcal{L}$.

Selected Short Answers:

1. $\mathbf{r} = 2\mathbf{i}+3\mathbf{j}+5\mathbf{k}+t(\mathbf{i}+3\mathbf{j}-\mathbf{k})$, $\left.\begin{array}{rcl} x &=& 2+t \\ y &=& 3+3t \\ z &=& 5-t \end{array}\right\} t \in \mathbb{R}$, $x-2 = \dfrac{y-3}{3} = \dfrac{z-5}{-1}$

2. $\mathbf{r} \cdot \big(\mathbf{i}+3\mathbf{j}-\mathbf{k}\big) = 6$, $x+3y-z = 6$.

3. (i) $-2\mathbf{i}-4\mathbf{j}-6\mathbf{k}$, $3\mathbf{i}-6\mathbf{j}+\mathbf{k}$ (ii) $-40\mathbf{i}-16\mathbf{j}+24\mathbf{k}$ (iii) $5x+2y-3z = 0$

4. (i) $\dfrac{x-1}{2} = \dfrac{y}{2} = \dfrac{z+1}{-1}$ (ii) $\dfrac{x+6}{2} = \dfrac{z-2}{-5}$, $y = 5$

(iii) $x = \dfrac{y-1}{2}$, $z = -1$ (iv) $y = 3$, $z = -3$

5. (i) $\dfrac{x+4}{2} = y-3 = \dfrac{z-5}{-6}$ (ii) $x = 5-y = -z$

6. (i) $3x+y-4z = 11$ (ii) $2x+z = 11$ (iii) $x+y-z = 9$ (iv) $y = 5$

7. $-7x+34y+36z = 124$

9. Point of intersection is $(-8, 4, -11)$. **10.** $\sqrt{21}/3$, $(5/3, -2/3, -2/3)$

11. (i) (b)(n) (ii) (d)(i) (iii) (f)(m) (iv) (a)(k) (v) (g)(l) (vi) (e)(h) (vii) (c)(j)

12. $\dfrac{x-1}{3} = \dfrac{y}{-4} = z+2$ **13.** $(2\mathbf{i}+3\mathbf{j}+4\mathbf{k}) \cdot (4\mathbf{i}+4\mathbf{j}-5\mathbf{k}) = 0$

14. $12x+9y+5z = 26$