Summer School | **Excercise Sheet 5 Solutions** | 2012

4. (i) $\mathbf{r} = \mathbf{i} - \mathbf{k} + t(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})\,,\qquad \left.\begin{array}{rcl} x &=& 1+2t \\ y &=& 2t \\ z &=& -1-t \end{array}\right\}\quad t \in \mathbb{R}\,,$

$$\frac{x-1}{2} = \frac{y}{2} = \frac{z+1}{-1}\,.$$

(ii) $\mathbf{r} = -6\mathbf{i} + 5\mathbf{j} + 2\mathbf{k} + t(2\mathbf{i} - 5\mathbf{k})\,,\qquad \left.\begin{array}{rcl} x &=& -6+2t \\ y &=& 5 \\ z &=& 2-5t \end{array}\right\}\quad t \in \mathbb{R}\,,$

$$\frac{x+6}{2} = \frac{z-2}{-5}\,,\quad y = 5\,.$$

(iii) $\mathbf{r} = \mathbf{j} - \mathbf{k} + t(\mathbf{i} + 2\mathbf{j})\,,\qquad \left.\begin{array}{rcl} x &=& t \\ y &=& 1+2t \\ z &=& -1 \end{array}\right\}\quad t \in \mathbb{R}\,,$

$$x = \frac{y-1}{2}\,,\quad z = -1\,.$$

(iv) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} + t\mathbf{i}\,,\qquad \left.\begin{array}{rcl} x &=& 2+t \\ y &=& 3 \\ z &=& -3 \end{array}\right\}\quad t \in \mathbb{R}\,,$

$$y = 3\,,\quad z = -3\,.$$

5. (i) $\mathbf{r} = -4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + t(2\mathbf{i} + \mathbf{j} - 6\mathbf{k})\,,\qquad \left.\begin{array}{rcl} x &=& -4+2t \\ y &=& 3+t \\ z &=& 5-6t \end{array}\right\}\quad t \in \mathbb{R}\,,$

$$\frac{x+4}{2} = y - 3 = \frac{z-5}{-6}\,.$$

(ii) $\mathbf{r} = 5\mathbf{j} + t(\mathbf{i} - \mathbf{j} - \mathbf{k})\,,\qquad \left.\begin{array}{rcl} x &=& t \\ y &=& 5-t \\ z &=& -t \end{array}\right\}\quad t \in \mathbb{R}\,,$

$$x = 5 - y = -z\,.$$

6. (i) $\big(\mathbf{r} - (4\mathbf{i} - \mathbf{j})\big) \cdot \big(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}\big) = 0$ or $\mathbf{r} \cdot \big(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}\big) = 11\,,$

$3x + y - 4z = 11.$

(ii) $\big(\mathbf{r} - (7\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})\big) \cdot \big(2\mathbf{i} + \mathbf{k}\big) = 0$ or $\mathbf{r} \cdot \big(2\mathbf{i} + \mathbf{k}\big) = 11$,

$2x + z = 11.$

(iii) $\big(\mathbf{r} + 9\mathbf{k}\big) \cdot \big(\mathbf{i} + \mathbf{j} - \mathbf{k}\big) = 0$ or $\mathbf{r} \cdot \big(\mathbf{i} + \mathbf{j} - \mathbf{k}\big) = 9$,

$x + y - z = 9.$

(iv) $\big(\mathbf{r} + (6\mathbf{i} - 5\mathbf{j} - 6\mathbf{k})\big) \cdot \mathbf{j} = 0$ or $\mathbf{r} \cdot \mathbf{j} = 5$,

$y = 5.$

7. Observe that

$$\overrightarrow{PQ} = -6\mathbf{i} - 15\mathbf{j} + 13\mathbf{k} \qquad \text{and} \qquad \overrightarrow{PR} = 8\mathbf{i} - 10\mathbf{j} + 11\mathbf{k}$$

so that

$$\overrightarrow{PQ} \times \overrightarrow{PR} = -35\mathbf{i} + 170\mathbf{j} + 180\mathbf{k} = 5(-7\mathbf{i} + 34\mathbf{j} + 36\mathbf{k}) \ .$$

Taking as normal vector $\mathbf{n} = -7\mathbf{i} + 34\mathbf{j} + 36\mathbf{k}$, and using the coordinates of Q, we get the Cartesian equation of the plane containing P, Q, R to be

$$-7x + 34y + 36z = 34(-8) + 36(11) = 124 \ .$$

8. (i) Setting $z = 0$ and solving $x + y = 2$ and $x - y = 0$ simultaneously yields the point $P(1, 1, 0)$ on $\mathcal{L}$.

(ii) Normals to the planes are $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ respectively, so that a vector pointing in the direction of $\mathcal{L}$ will be

$$\big(\mathbf{i} + \mathbf{j} + \mathbf{k}\big) \times \big(\mathbf{i} - \mathbf{j} + 3\mathbf{k}\big) = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \ .$$

(iii) Hence parametric scalar equations for $\mathcal{L}$ are

$$\left.\begin{array}{rcr} x &=& 1 + 4t \\ y &=& 1 - 2t \\ z &=& -2t \end{array}\right\} \quad t \in \mathbb{R} \ ,$$

and Cartesian equations for $\mathcal{L}$ are

$$\frac{x-1}{4} = \frac{y-1}{-2} = \frac{z}{-2} \ .$$

9. The lines intersect if and only if there exist t and s such that

$$\mathbf{i} + \mathbf{j} + \mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 6\mathbf{i} - 6\mathbf{j} + \mathbf{k} + s(-7\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}) \ .$$

Equating coefficients this is equivalent to

$$\left\{\begin{array}{rcr} 1 + 3t &=& 6 - 7s \\ 1 - t &=& -6 + 5s \\ 1 + 4t &=& 1 - 6s \end{array}\right. ,$$

that is,

$$\begin{cases} 3t + 7s &= 5 \\ t + 5s &= 7 \\ 4t + 6s &= 0 \end{cases} .$$

Solving simultaneously yields a solution $s = 2$ and $t = -3$, which correspond to the point of intersection $(-8, 4, -11)$.

10.* The point $Q(0, 3, 0)$ lies on $\mathcal{P}$, and a normal vector is $\mathbf{n} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The distance from P to $\mathcal{P}$ is

$$\left|\overrightarrow{PQ} \cdot \widehat{\mathbf{n}}\right| = \frac{\left|(-3\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \cdot (4\mathbf{i} + 2\mathbf{j} - \mathbf{k})\right|}{\sqrt{16 + 4 + 1}} = \frac{|-7|}{\sqrt{21}} = \frac{7}{\sqrt{21}} = \frac{\sqrt{21}}{3} .$$

Denote the closest point of $\mathcal{P}$ to P by R. Observe that $\overrightarrow{PQ} \cdot \mathbf{n} < 0$, so that

$$\overrightarrow{PR} = -\frac{\sqrt{21}}{3}\widehat{\mathbf{n}} = -\frac{1}{3}(4\mathbf{i} + 2\mathbf{j} - \mathbf{k}) .$$

Hence

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = 3\mathbf{i} - \mathbf{k} - \frac{1}{3}(4\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \frac{5}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} ,$$

so that

$$R = (5/3, -2/3, -2/3) .$$

11. (i) (b)(n) (ii) (d)(i) (iii) (f)(m) (iv) (a)(k) (v) (g)(l) (vi) (e)(h) (vii) (c)(j)

12. The line contains $(1, 0, -2)$ and has direction $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, so has parametric equations

$$\left.\begin{aligned} x &= 1 + 3t \\ y &= -4t \\ z &= -2 + t \end{aligned}\right\} \quad t \in \mathbb{R} ,$$

yielding Cartesian equations

$$\frac{x - 1}{3} = \frac{y}{-4} = z + 2 .$$

13. The line has direction $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and the plane has normal $\mathbf{n} = 4\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$, so it is sufficient to check that $\mathbf{v} \cdot \mathbf{n} = 0$, which is indeed the case:

$$\mathbf{v} \cdot \mathbf{n} = 2(4) + 3(4) + 4(-5) = 8 + 12 - 20 = 0 .$$

14. The plane contains $P(1, 1, 1)$ and $Q(4, -3, 1)$ and has a parallel vector in the direction of the line which is

$$\mathbf{v} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k} .$$

A normal to the plane therefore is

$$\overrightarrow{PQ} \times \mathbf{v} = (3\mathbf{i} - 4\mathbf{j}) \times (-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = -12\mathbf{i} - 9\mathbf{j} - 5\mathbf{k} .$$

Hence the Cartesian equation is

$$12x + 9y + 5z \;=\; 12 + 9 + 5 \;=\; 26\,.$$

15.* The point $Q(1, 1, -4)$ lies on $\mathcal{L}$, which has direction $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Hence the distance from P to $\mathcal{L}$ is

$$\frac{\left|\overrightarrow{PQ} \times \mathbf{v}\right|}{|\mathbf{v}|} \;=\; \frac{\left|(-\mathbf{i} - 5\mathbf{k}) \times (\mathbf{i} + 3\mathbf{j} - \mathbf{k})\right|}{\sqrt{11}} \;=\; \frac{|15\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}|}{\sqrt{11}} \;=\; \frac{\sqrt{270}}{\sqrt{11}}\,.$$

Let R be the closest point on $\mathcal{L}$ to P, so $\left|\overrightarrow{PR}\right| = \dfrac{\sqrt{270}}{\sqrt{11}}$. By Pythagoras,

$$\left|\overrightarrow{QR}\right| \;=\; \sqrt{\left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{PR}\right|^2} \;=\; \sqrt{26 - \frac{270}{11}} \;=\; \frac{4}{\sqrt{11}}\,.$$

Note that

$$\overrightarrow{PQ} \cdot \mathbf{v} \;=\; 4 > 0\,,$$

so

$$\overrightarrow{OR} \;=\; \overrightarrow{OQ} + \overrightarrow{QR} \;=\; \overrightarrow{OQ} - \frac{4}{\sqrt{11}}\,\widehat{\mathbf{v}}$$

$$=\; \mathbf{i} + \mathbf{j} - 4\mathbf{k} - \frac{4}{11}(\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$

$$=\; \frac{1}{11}\big(7\mathbf{i} - \mathbf{j} - 40\mathbf{k}\big)\,.$$

Hence $R = (7/11, -1/11, -40/11)$.