

*Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.*

**Important Ideas and Useful Facts:**

- (i) A *linear equation* in variables  $x_1, x_2, \dots, x_n$  has the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where  $a_1, a_2, \dots, a_n, b$  are constants.

- (ii) A *system* of linear equations has the form

$$\begin{array}{cccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

with *augmented matrix*

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right].$$

The system is *homogeneous* if  $b_1 = b_2 = \dots = b_m = 0$ .

- (iii) Every system of linear equations has either no solutions, one solution, or infinitely many solutions. If it has no solutions then the system is called *inconsistent*. If it has at least one solution then the system is called *consistent*.
- (iv) There are three types of *elementary row operations* performed on augmented matrices:
- (a) interchanging the  $i$ th and  $j$ th rows (denoted by  $R_i \leftrightarrow R_j$ )
  - (b) multiplying the  $i$ th row through by a nonzero constant  $\lambda$  (denoted by  $R_i \rightarrow \lambda R_i$ )
  - (c) adding a multiple of the  $j$ th row to the  $i$ th row (denoted by  $R_i \rightarrow R_i + \lambda R_j$ )
- (v) A matrix is in *row echelon form* if
- (a) rows of zeros appear at the bottom,
  - (b) first nonzero (*leading*) entries of consecutive rows appear further to the right,
  - (c) leading entries of rows are equal to 1;
- and in *reduced row echelon form* if, in addition,
- (d) entries above (and below) leading entries are zero.

- (vi) The process of *Gaussian elimination* applies elementary row operations (*row reduction*) to transform the augmented matrix of a system into row echelon form, after which the associated system is solved using *back substitution*:
- the *leading variables* corresponding to leading entries are evaluated one equation at a time from the bottom towards the top,
  - parameters are assigned to each nonleading variable (if any).
- (vii) A system is inconsistent if and only if at some stage in the process of row reduction a row of the form
- $$0 \quad 0 \quad \cdots \quad 0 \quad | \quad k$$
- is produced for some nonzero real number  $k$ .
- (viii) The process of *Gauss-Jordan elimination* row reduces the augmented matrix to reduced row echelon form, after which the process of back substitution simplifies. However, Gauss-Jordan elimination is usually less efficient in terms of the overall number of arithmetic operations used than Gaussian elimination.

### Preparatory Exercises:

1. Solve the following systems of equations:

$$\begin{array}{ll} \text{(i)} & \begin{array}{l} x + y = 6 \\ 2x - 3y = 2 \end{array} \\ \text{(ii)} & \begin{array}{l} x + 2y + 3z = 6 \\ y - z = 3 \\ 2z = -4 \end{array} \end{array}$$

2. Solve the following systems of equations by writing down the associated augmented matrix and row reducing, either to echelon form and using back substitution, or to reduced echelon form and reading off the answers:

$$\begin{array}{ll} \text{(i)} & \begin{array}{l} x + y - z = 0 \\ 2x - y + z = 9 \\ x + z = 10 \end{array} \\ \text{(ii)} & \begin{array}{l} -3x + 2y + z = 4 \\ 4x + y + 3z = 9 \\ x - y - z = -4 \end{array} \end{array}$$

3. Solve the following systems of equations by assigning  $z = t$  and expressing  $x$  and  $y$  in terms of the parameter  $t$ :

$$\begin{array}{ll} \text{(i)} & \begin{array}{l} x - 2z = 4 \\ y + z = 2 \end{array} \\ \text{(ii)} & \begin{array}{l} x + 2y + 3z = 0 \\ y - 2z = -1 \end{array} \end{array}$$

### Tutorial Exercises:

4. Solve the following systems of equations:

$$\begin{array}{ll} \text{(i)} & \begin{array}{l} 4x - 5y = 7 \\ -3x + 8y = -1 \end{array} \\ \text{(ii)} & \begin{array}{l} x + 2y + z = 1 \\ -x + y + 2z = 2 \\ 2x + 3y + 2z = 5 \end{array} \end{array}$$

5. Find parametric scalar equations for the line of intersection of the two planes in each of the following cases:

$$\begin{array}{ll} \text{(i)} & \begin{array}{l} x + y + z = 2 \\ x - y + 3z = 0 \end{array} \\ \text{(ii)} & \begin{array}{l} -3x + 2y + 7z = 1 \\ 5x - 3y - 2z = -2 \end{array} \end{array}$$

6. Solve the following homogeneous systems of equations:

$$\begin{array}{ll} \text{(i)} & \begin{array}{l} x + 2y + 3z = 0 \\ 3x + 2y + z = 0 \end{array} \\ \text{(ii)} & \begin{array}{l} -x + y + z - w = 0 \\ 2x + z + w = 0 \\ x - 2y + z + 3w = 0 \end{array} \end{array}$$

7. Give a very brief reason why a homogeneous system can never be inconsistent.

8. Solve the following systems of equations:

$$\begin{array}{ll} \text{(i)} & \begin{array}{l} x + 2y + 7z = 5 \\ x + y + 4z = 3 \\ 2x + 3y + 11z = 7 \end{array} \\ \text{(ii)} & \begin{array}{l} x + 2y + z - w = 4 \\ 2x + 4y - z + 4w = -1 \\ -x - 2y + 2z - 5w = 5 \end{array} \end{array}$$

9. (suitable for group discussion) A cubic polynomial in  $x$  takes the value  $-2$  when  $x = 1$  and the value  $-10$  when  $x = -1$ . Its derivative takes the value  $0$  when  $x = 1$  and the value  $12$  when  $x = -1$ . Find the polynomial.

10.\* Find the values of  $\lambda$  such that the following system (i) is inconsistent; (ii) has infinitely many solutions; (iii) has a unique solution:

$$\begin{array}{rcl} x & - & 3z = -3 \\ -2x - \lambda y + z & = & 2 \\ x + 2y + \lambda z & = & 1 \end{array}$$

### Further Exercises:

11. For each of the following augmented matrices, decide whether the system of equations to which it corresponds has (a) no solution, (b) a unique solution, or (c) an infinite solution.

$$\begin{array}{ll} \text{(i)} & \left[ \begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 5 \\ -1 & -4 & 0 & 1 \end{array} \right] \\ \text{(ii)} & \left[ \begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 5 \\ -1 & -4 & 0 & -1 \end{array} \right] \\ \text{(iii)} & \left[ \begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ -1 & -2 & 3 & 6 \\ -1 & -3 & 0 & 1 \end{array} \right] \\ \text{(iv)} & \left[ \begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 1 & -1 \\ -2 & -2 & 4 & 6 & 2 & 0 \\ 0 & 0 & 0 & -3 & -1 & 4 \end{array} \right] \end{array}$$

12. The Jones family consists of Ann and Bill and their two children Cathy and Daniel. Their current ages add up to 70. Bill is three times as old as the present total age of Cathy and Daniel. In 10 years time, Ann's age will be 20 less than twice the total of the then ages of Cathy and Daniel. Four years ago, Cathy's age was equal to Ann's age minus Bill's age. Find the present ages of Ann, Bill, Cathy and Daniel.

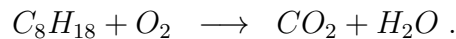
13. Solve the following homogeneous system of equations:

$$\begin{array}{rcl} 2x_1 + 3x_2 + x_3 - x_4 + 4x_5 & = & 0 \\ -2x_1 - 3x_2 + x_3 + 2x_4 - 3x_5 & = & 0 \\ 2x_1 + 3x_2 + 2x_3 + 2x_4 + 2x_5 & = & 0 \end{array}$$

14.\* Find  $A$ ,  $B$ ,  $C$  and  $D$  such that

$$\frac{x^3}{(x-1)^4} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4}.$$

15.\* The combustion of petrol is described by the chemical reaction



Find the smallest positive integer values for  $x$ ,  $y$ ,  $z$  and  $w$  such that



is balanced, in the sense that the number of atoms of each type agree on the left and right hand sides.

### Short Answers:

1. (i)  $x = 4$ ,  $y = 2$  (ii)  $x = 10$ ,  $y = 1$ ,  $z = -2$
2. (i)  $x = 3$ ,  $y = 4$ ,  $z = 7$  (ii)  $x = -1$ ,  $y = -2$ ,  $z = 5$
3. (i)  $x = 4 + 2t$ ,  $y = 2 - t$ ,  $z = t$  (ii)  $x = 2 - 7t$ ,  $y = -1 + 2t$ ,  $z = t$
4. (i)  $x = 3$ ,  $y = 1$  (ii)  $x = 3$ ,  $y = -3$ ,  $z = 4$
5. (i)  $x = 1 - 2t$ ,  $y = 1 + t$ ,  $z = t$  (ii)  $x = -1 - 17t$ ,  $y = -1 - 29t$ ,  $z = t$
6. (i)  $x = t$ ,  $y = -2t$ ,  $z = t$  (ii)  $x = -\frac{2}{7}t$ ,  $y = \frac{8}{7}t$ ,  $z = -\frac{3}{7}t$ ,  $w = t$
7. Assigning each variable the value 0 yields at least one solution.
8. (i) inconsistent, no solution (ii)  $x = 1 - 2s - t$ ,  $y = s$ ,  $z = 3 + 2t$ ,  $w = t$
9.  $x^3 - 3x^2 + 3x - 3$
10. (i)  $\lambda = -5$  (ii)  $\lambda = 2$  (iii)  $\lambda \neq 2, -5$
11. (i) (c) (ii) (a) (iii) (b) (iv) (a)
12.  $A = 30$ ,  $B = 30$ ,  $C = 4$ ,  $D = 6$
13.  $x_1 = -\frac{3}{2}s - t$ ,  $x_2 = s$ ,  $x_3 = -t$ ,  $x_4 = t$ ,  $x_5 = t$
14.  $A = 1$ ,  $B = 3$ ,  $C = 3$ ,  $D = 1$
15.  $x = 2$ ,  $y = 25$ ,  $z = 16$ ,  $w = 18$