Summer School **Excercise Sheet 6 Solutions** 2012

1. (i) $\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 2 & -3 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & -5 & -10 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right]$,

so that $x = 4$ and $y = 2$.

(ii) By back substitution, $z = -2$, $y = 3 + z = 1$, $x = 6 - 2y - 3z = 10$.

2. (i) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 9 \\ 1 & 0 & 1 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 9 \\ 0 & -1 & 2 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 7 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 7 \end{array}\right]$, so that $x = 3$, $y = 4$, $z = 7$.

(ii) $\left[\begin{array}{ccc|c} -3 & 2 & 1 & 4 \\ 4 & 1 & 3 & 9 \\ 1 & -1 & -1 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & -4 \\ 0 & -1 & -2 & -8 \\ 0 & 5 & 7 & 25 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & -3 & -15 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$, so that $x = -1$, $y = -2$, $z = 5$.

3. By back substitution: (i) $z = t$, $y = 2 - z = 2 - t$, $x = 4 + 2z = 4 + 2t$;

(ii) $z = t$, $y = -1 + 2z = -1 + 2t$, $x = -2y - 3z = -2(-1 + 2t) - 3t = 2 - 7t$.

4. (i) $\left[\begin{array}{cc|c} 4 & -5 & 7 \\ -3 & 8 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -5 & 7 \\ 1 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & -17 & -17 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$,

so that $x = 3$ and $y = 1$.

(ii) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 2 \\ 2 & 3 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & -1 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 3 & 12 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$, so that $x = 3$, $y = -3$, $z = 4$.

5. (i) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right]$,

so that, by back substitution, $z = t$, $y = 1 + t$, $x = 1 - 2t$.

(ii) $\left[\begin{array}{ccc|c} -3 & 2 & 7 & 1 \\ 5 & -3 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 6 & -4 & -14 & -2 \\ 5 & -3 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -12 & 0 \\ 5 & -3 & -2 & -2 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & -12 & 0 \\ 0 & 2 & 58 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 17 & -1 \\ 0 & 1 & 29 & -1 \end{array}\right]$,

so that, by back substitution, $z = t$, $y = -1 - 29t$, $x = -1 - 17t$.

6. (i) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$,

so that, by back substitution, $z = t,\ y = -2t\,,\ x = t\,.$

(ii) $\left[\begin{array}{cccc|c} -1 & 1 & 1 & -1 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & -2 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 \\ 0 & -1 & 2 & 2 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 0 & -3 & -1 & 0 \\ 0 & 1 & -2 & -2 & 0 \\ 0 & 0 & 7 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{7} & 0 \\ 0 & 1 & 0 & -\frac{8}{7} & 0 \\ 0 & 0 & 1 & \frac{3}{7} & 0 \end{array}\right]$,

so that, by back substitution, $w = t,\ z = -\frac{3}{7}t\,,\ y = \frac{8}{7}t\,,\ x = -\frac{2}{7}t\,.$

7. If we assign zero to each variable then each equation is satisfied, so that there is at least one solution of any homogeneous system. Hence all homogeneous systems are consistent.

8. (i) $\left[\begin{array}{ccc|c} 1 & 2 & 7 & 5 \\ 1 & 1 & 4 & 3 \\ 2 & 3 & 11 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 7 & 5 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 7 & 5 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right]$,

so that the system is inconsistent, that is, has no solution.

(ii) $\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 2 & 4 & -1 & 4 & -1 \\ -1 & -2 & 2 & -5 & 5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 0 & 0 & -3 & 6 & -9 \\ 0 & 0 & 3 & -6 & 9 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$,

so that, by back substitution, $w = t\,,\ z = 3 + 2t\,,\ y = s\,,\ x = 1 - 2s - t\,.$

9. Call the polynomial

$$p(x) = ax^3 + bx^2 + cx + d$$

where a, b, c, d are constants to be determined, so that $p'(x) = 3ax^2 + 2bx + c\,.$ But

$$p(1) = -2\,, \quad p(-1) = -10\,, \quad p'(1) = 0\,, \quad p'(-1) = 12\,,$$

yielding the system

$$\begin{array}{rcrcrcrcr} a & + & b & + & c & + & d & = & -2 \\ -a & + & b & - & c & + & d & = & -10 \\ 3\,a & + & 2\,b & + & c & & & = & 0 \\ 3\,a & - & 2\,b & + & c & & & = & 12 \end{array}$$

and augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ -1 & 1 & -1 & 1 & -10 \\ 3 & 2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 12 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ 0 & 2 & 0 & 2 & -12 \\ 0 & -1 & -2 & -3 & 6 \\ 0 & -5 & -2 & -3 & 18 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 1 & -6 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & 2 & -12 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 & -6 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 & -12 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & -6 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right],$$

yielding $a = 1$, $b = -3$, $c = 3$, $d = -3$, so that $p(x) = x^3 - 3x^2 + 3x - 3$.

10.* $$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ -2 & -\lambda & 1 & 2 \\ 1 & 2 & \lambda & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & -\lambda & -5 & -4 \\ 0 & 2 & \lambda+3 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & \frac{\lambda+3}{2} & 2 \\ 0 & -\lambda & -5 & -4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & \frac{\lambda+3}{2} & 2 \\ 0 & 0 & -5 + \frac{\lambda(\lambda+3)}{2} & -4 + 2\lambda \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 2 & \lambda+3 & 4 \\ 0 & 0 & (\lambda+5)(\lambda-2) & 4(\lambda-2) \end{array}\right]$$

(i) To be inconsistent, we require $(\lambda+5)(\lambda-2) = 0$ and $4(\lambda-2) \neq 0$, so that $\lambda = -5$.

(ii) To have infinitely many solutions, we require $(\lambda+5)(\lambda-2) = 0 = 4(\lambda-2)$, so that $\lambda = 2$.

(iii) To have a unique solution, we require both (i) and (ii) to fail, that is, $\lambda \neq 2, -5$.

11. (i) $$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 5 \\ -1 & -4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which has an infinite solution, using one parameter.

(ii) $$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 5 \\ -1 & -4 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right],$$

which is inconsistent, that is, has no solution.

(iii) $$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ -1 & -2 & 3 & 6 \\ -1 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right],$$

which has a unique solution.

(iv) $$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 1 & -1 \\ -2 & -2 & 4 & 6 & 2 & 0 \\ 0 & 0 & 0 & -3 & -1 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 1 & -1 \\ 0 & 0 & 0 & 12 & 4 & -2 \\ 0 & 0 & 0 & -3 & -1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 1 & -1 \\ 0 & 0 & 0 & 3 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 14 \end{array}\right],$$

which is inconsistent, that is, has no solution.

12. If we call the ages of the family members A, B, C, D respectively, then the information tells us that $A+B+C+D = 70$, $B = 3(C+D)$, $A+10 = 2((C+10)+(D+10))-20$, $C - 4 = (A-4) - (B-4)$. This becomes the following system of equations:

$$\begin{array}{ccccccccc} A & + & B & + & C & + & D & = & 70 \\ & & B & - & 3C & - & 3D & = & 0 \\ A & & & - & 2C & - & 2D & = & 10 \\ A & - & B & - & C & & & = & -4 \end{array}$$

with augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 70 \\ 0 & 1 & -3 & -3 & 0 \\ 1 & 0 & -2 & -2 & 10 \\ 1 & -1 & -1 & 0 & -4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 4 & 4 & 70 \\ 0 & 1 & -3 & -3 & 0 \\ 0 & -1 & -3 & -3 & -60 \\ 0 & -2 & -2 & -1 & -74 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 4 & 4 & 70 \\ 0 & 1 & -3 & -3 & 0 \\ 0 & 0 & -6 & -6 & -60 \\ 0 & 0 & -8 & -7 & -74 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 0 & 30 \\ 0 & 0 & 1 & 1 & 10 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 0 & 30 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right],$$

yielding $A = 30$, $B = 30$, $C = 4$, $D = 6$.

13. $$\begin{bmatrix} 2 & 3 & 1 & -1 & 4 \\ -2 & -3 & 1 & 2 & -3 \\ 2 & 3 & 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 1 & -1 & 4 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & 0 & -4 & 6 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

so that, by back substitution, $x_5 = t$, $x_4 = t$, $x_3 = -t$, $x_2 = s$, $x_1 = -\frac{3}{2}s - t$.

14.* We have $x^3 = A(x-1)^3 + B(x-1)^2 + C(x-1) + D$ for all x (by continuity). Putting $x = 1$ gives $D = 1$ immediately. Putting $x = 2,\ 0,\ -1$ respectively yields the system

$$\begin{array}{rcrcrcr} A & + & B & + & C & = & 7 \\ -A & + & B & - & C & = & -1 \\ -8A & + & 4B & - & 2C & = & -2 \end{array}$$

with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ -1 & 1 & -1 & -1 \\ -8 & 4 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 2 & 0 & 6 \\ 0 & 12 & 6 & 54 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 6 & 18 \end{array}\right],$$

yielding $A = 1$, $B = 3$, $C = 3$, $D = 1$.

15.* Counting carbon, hydrogen and oxygen atoms we get the homogeneous system

$$\begin{array}{rcrcrcrcr} 8x & & & - & z & & & = & 0 \\ 18x & & & & & - & 2w & = & 0 \\ & & 2y & - & 2z & - & w & = & 0 \end{array}$$

with coefficient matrix $\begin{bmatrix} 8 & 0 & -1 & 0 \\ 18 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/8 & 0 \\ 0 & 0 & 9/4 & -2 \\ 0 & 1 & -1 & -1/2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & -1/8 & 0 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 1 & -8/9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -25/18 \\ 0 & 0 & 1 & -8/9 \end{bmatrix},$$

with parametric solution $x = t/9$, $y = 25t/18$, $z = 8t/9$, $w = t$, having smallest positive integer solution $x = 2$, $y = 25$, $z = 16$, $w = 18$, yielding

$$2\,C_8H_{18} + 25\,O_2 \longrightarrow 16\,CO_2 + 18\,H_2O\,.$$