The University of Sydney

MATH1002 Linear Algebra

Summer School	**Exercise Sheet 7**	2012

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) A *matrix* is an array of numbers, called *entries*. The plural of matrix is *matrices*. If a matrix M has m rows and n columns then we say that M is $m \times n$. We call a matrix M *square* if M is $n \times n$ for some n. The entry lying in the ith row and jth column is called the (i, j)-*entry*.

(ii) A matrix consisting of one row is called a *row vector*. A matrix consisting of one column is called a *column vector*.

(iii) To *add* or *subtract* using two matrices of the same size, simply add or subtract the corresponding entries. To form the *negative* of a matrix, simply take the negatives of its entries. To *multiply a matrix by a scalar*, simply multiply its entries by the scalar.

(iv) The *zero matrix* has all of its entries equal to 0, and is denoted by 0 or $0_{m \times n}$ if the size needs to be emphasised.

(v) The *identity matrix* is a square matrix with *diagonal* entries equal to 1 and all entries off the diagonal equal to 0. The identity matrix is denoted by I or I_n if it is $n \times n$ and the size needs to be emphasised.

(vi) If A is an $m \times n$ matrix and B is an $n \times p$ matrix then the *matrix product* AB is defined and is an $m \times p$ matrix. The (i, k)-entry of AB is the "dot product" of the ith row of A with the kth column of B.

(vii) If A, B, C are matrices of appropriate sizes for which the expressions make sense, and λ and μ are scalars, then the following properties hold:

$$A + B = B + A\,, \qquad (A + B) + C = A + (B + C)\,, \qquad A + 0 = 0 + A = A\,,$$

$$-(-A) = A\,, \qquad A + (-A) = A - A = 0\,, \qquad \lambda(\mu A) = (\lambda\mu)A\,,$$

$$\lambda(A + B) = \lambda A + \lambda B\,, \qquad (\lambda + \mu)A = \lambda A + \mu A\,, \qquad IA = AI = A\,,$$

$$(AB)C = A(BC)\,, \qquad A(B + C) = AB + AC\,, \qquad (A + B)C = AC + BC\,,$$

$$\lambda(BC) = (\lambda B)C = B(\lambda C)\,, \qquad 0A = 0 = A0\,.$$

(viii) Warning: Matrix multiplication is not in general commutative. Most of the time

$$AB \neq BA\,.$$

Preparatory Exercises:

1. Let $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 & 0 \\ -1 & -2 & 6 \end{bmatrix}$, $C = \begin{bmatrix} -3 & 5 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 4 \\ 2 & 0 \\ 0 & -7 \\ 1 & -3 \end{bmatrix}$.

Write down the sizes of A, B, C and D.

2. For the matrix

$$M = \begin{bmatrix} 6 & 0 & 3 & -5 \\ 0 & 7 & 2 & 4 \\ 1 & 3 & -2 & 0 \end{bmatrix},$$

locate the

(i) $(2,2)$-entry (ii) $(3,3)$-entry (iii) $(1,4)$-entry (iv) $(3,2)$-entry (v) $(3,4)$-entry

3. Consider the following 2×2 matrices:

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 2 \\ 4 & -3 \end{bmatrix}, \; D = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

Find

(i) $A+B$ (ii) $A-B$ (iii) $B-C$ (iv) $D+C$ (v) $2A$ (vi) $-B$ (vii) $\frac{1}{5}D$
(viii) AB (ix) BA (x) CD (xi) BC (xii) $A(BC)$ (xiii) $(AB)C$
(xiv) $ABCD$ (xv) A^2 (xvi) B^2 (xvii) $A^2 - B^2$ (xviii) $(A+B)(A-B)$

Tutorial Exercises:

4. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}, \; C = \begin{bmatrix} 3 & 4 & 2 \end{bmatrix}, \; D = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}.$$

(i) Find AB, BA, CD, DC and $BA + DC$.

(ii) Explain briefly why A^2, B^2, C^2, D^2 and $AB + CD$ do not exist.

5. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$.

(i) Find AB, BA, CD and DC.

(ii) Simplify A^2B^2 and $C(DCDCD)^2C$ without any further matrix calculations.

6. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $Y = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $Z = \begin{bmatrix} 5 & -2 \end{bmatrix}$, $W = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

Find AX, AY, ZA, WA, ZX, ZAX, ZY, ZAY, WX, WAX, WY, WAY.

7. Find a 2×2 matrix M such that $M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but every entry of M is nonzero.

8. Explain briefly why the associative law for matrix multiplication implies that every square matrix commutes with its square.

9. (suitable for group discussion) Which of the following do you know to be true or expect to be true for all square matrices A, B, C of the same size:

(i)	$(AB)C = A(BC)$	(ii)	$AB = BA$
(iii)	$(AB)^2 = A^2B^2$	(iv)	$A(B+C) = AB + AC$
(v)	$(-A)(-B) = AB$	(vi)	$A(B-C) = AB - AC$
(vii)	$(A+B)^2 = A^2 + 2AB + B^2$	(viii)	$(A+B)(A-B) = A^2 - B^2$
(ix)	$(A+I)^2 = A^2 + 2A + I$	(x)	$(A+I)(A-I) = A^2 - I$

Find a counterexample to each statement that you believe not to be true in general.

10.* Consider the matrix

$$M = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} .$$

(i) Verify that $M^2 = 2M - I$.

(ii) Deduce that $M^3 = 3M - 2I$ and guess a general formula for powers of M. (If you know the technique of proof by induction then you can try to prove that your guess is correct.)

(iii) Evaluate M^5, M^{10} and M^{100}.

Further Exercises:

11. *(Homework)* Solve each of the following for x, y and z:

(i) $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 2 \\ 0 & 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 29 \\ -10 \end{bmatrix}$$

(ii) $$\begin{bmatrix} 2 & -3 & 3 \\ 4 & 9 & -4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

12. A square matrix is called *diagonal* if all entries away from the main diagonal are zero. Find a simple rule for multiplying diagonal matrices. More generally, describe in words as simply as you can what happens if you multiply any square matrix (of the same size) (i) on the left by a diagonal matrix, or (ii) on the right by a diagonal matrix.

13. Find all x, y, z and w such that the following matrix equation holds:

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

14.* Find XY in each case, given that X is a row matrix and Y is a column matrix, both with the same number of entries:

(i) $$YX = \begin{bmatrix} -2 & -3 \\ 2 & 3 \end{bmatrix}$$

(ii) $$YX = \begin{bmatrix} 3 & -3 & 6 \\ 4 & -4 & 8 \\ -2 & 2 & -4 \end{bmatrix}$$

15.* *(Homework)* Find necessary and sufficient conditions on a, b, c and d such that the matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with the matrix A in each of the following cases:

(i) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 & 7 \\ 7 & 0 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Short Answers:

1. $2 \times 2\,,\ 2 \times 3\,,\ 1 \times 3\,,\ 4 \times 2$ **2.** (i) 7 (ii) -2 (iii) -5 (iv) 3 (v) 0

3. (i) $\begin{bmatrix} 4 & 2 \\ -2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & -6 \\ 0 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 10 & 2 \\ 4 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & -4 \\ -2 & 6 \end{bmatrix}$
(vi) $\begin{bmatrix} -3 & -4 \\ 1 & 2 \end{bmatrix}$ (vii) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (viii) $\begin{bmatrix} 5 & 8 \\ -6 & -10 \end{bmatrix}$ (ix) $\begin{bmatrix} -1 & 6 \\ 1 & -4 \end{bmatrix}$ (x) $\begin{bmatrix} 0 & 10 \\ 40 & -15 \end{bmatrix}$
(xi) $\begin{bmatrix} 16 & -6 \\ -8 & 4 \end{bmatrix}$ (xii) (xiii) $\begin{bmatrix} 32 & -14 \\ -40 & 18 \end{bmatrix}$ (xiv) $\begin{bmatrix} 320 & -70 \\ -400 & 90 \end{bmatrix}$ (xv) $\begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$
(xvi) $\begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix}$ (xvii) $\begin{bmatrix} -2 & -12 \\ -3 & 11 \end{bmatrix}$ (xviii) $\begin{bmatrix} -8 & -14 \\ 4 & 17 \end{bmatrix}$

4. (i) $\begin{bmatrix} -1 & 5 \\ 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 & 1 \\ -2 & 0 & 2 \\ 5 & 2 & -1 \end{bmatrix}$, $[\,3\,] = 3$, $\begin{bmatrix} -3 & -4 & -2 \\ -3 & -4 & -2 \\ 15 & 20 & 10 \end{bmatrix}$, $\begin{bmatrix} 0 & -2 & -1 \\ -5 & -4 & 0 \\ 20 & 22 & 9 \end{bmatrix}$
(ii) None of A, B, C, D are square. The matrices AB and CD have different sizes.

5. (i) I_2, I_2, I_3, I_3 (ii) I_2, I_3 **6.** $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 12 \\ 30 \end{bmatrix}$, $\begin{bmatrix} -5 & 2 \end{bmatrix}$, $\begin{bmatrix} 6 & 6 \end{bmatrix}$, 7, -7, 0, 0, 0, 0, 7, 42

7. $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ **8.** $AA^2 = A(AA) = (AA)A = A^2A$ **9.** (i), (iv), (v), (vi), (ix), (x)

10. (iii) $\begin{bmatrix} 11 & -5 \\ 20 & -9 \end{bmatrix}$, $\begin{bmatrix} 21 & -10 \\ 40 & -19 \end{bmatrix}$, $\begin{bmatrix} 201 & -100 \\ 400 & -199 \end{bmatrix}$

12. Multiply corresponding diagonal elements. (i) Multiply each row by the corresponding diagonal element. (ii) Multiply each column by the corresponding diagonal element.

13. $x = s\,,\ y = -4t\,,\ z = s\,,\ w = t$ **14.** (i) 1 (ii) -5