Summer School | **Exercise Sheet 9** | 2012

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) The *determinant* of a 1×1 matrix $[a]$ is simply the entry a.

(ii) The *determinant* of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

(iii) The *determinant* of a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ is

$$\det A = |A| = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

called the *expansion along the first row*, where the smaller determinant arises by ignoring the row and column of the entry being used as a coefficient.

(iv) Expanding along any row or down any column of a square matrix A produces the same real number, called the *determinant* of A, denoted by $\det A$ or $|A|$, provided one uses adjustment factors given by the chequerboard patterns

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

and so on to higher dimensions.

(v) Determinant method for cross products: If $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ and $\mathbf{w} = d\,\mathbf{i} + e\,\mathbf{j} + f\,\mathbf{k}$ then

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} .$$

(vi) Multiplicative property: $\det(AB) = (\det A)(\det B)$.

(vii) Invertibility criterion: A square matrix is invertible if and only if its determinant is nonzero.

(viii) If B is obtained from A by swapping two rows or swapping two columns then

$$\det B = -\det A .$$

(ix) If B is obtained from A by multiplying a row or column by λ then

$$\det B = \lambda \det A .$$

(x) If B is obtained from A by adding a multiple of one row [column] to another row [column] then

$$\det B = \det A .$$

(xi) If B is the *transpose* of A, that is, obtained by interchanging rows and columns, then

$$\det B = \det A .$$

(xii) If A is *triangular*, that is all entries above or below the diagonal are zero, then $\det A$ is the product of the diagonal elements.

Preparatory Exercises:

1. Find the following determinants:

$$\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix}, \quad \begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}, \quad \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 5 & 4 \\ 3 & 3 \end{vmatrix} .$$

2. Find the determinant $\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix}$ by expanding along the first row.

3. Now find the determinant of the previous exercise by expanding

(i) along the second row (ii) along the third row (iii) down the third column

Tutorial Exercises:

4. Find the following determinants:

(i) $\begin{vmatrix} 5 & 2 \\ 3 & -2 \end{vmatrix}$ (ii) $\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix}$

(v) $\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ -1 & 1 & 0 \end{vmatrix}$ (vi) $\begin{vmatrix} 2 & 4 & 6 \\ 7 & 11 & 6 \\ -6 & -6 & 12 \end{vmatrix}$ (vii) $\begin{vmatrix} -4 & 3 & 3 \\ 8 & 7 & 3 \\ 4 & 3 & 3 \end{vmatrix}$

5. Write down quickly the determinants of the following matrices:

(i) $\begin{bmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -5 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 3 & 8 \\ 0 & -6 & -7 \\ 0 & 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} -4 & -5 & 11 \\ 0 & 0 & 0 \\ 2 & -1 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 0 & 5 \\ 6 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ -6 & -3 & -7 & -1 \end{bmatrix}$

6. Justify briefly the following calculation:

$$\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = \begin{vmatrix} 2 & -3 & -2 \\ 3 & -3 & 0 \\ 1 & -14 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 & -2 \\ 3 & 0 & 0 \\ 1 & -13 & 0 \end{vmatrix} = -2\begin{vmatrix} 3 & 0 \\ 1 & -13 \end{vmatrix} = 78$$

7. Use elementary row and column operations, or otherwise, to find the following:

(i) $\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 4 & 5 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{vmatrix}$

8. Find $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ in each of the following cases:

(i) $\mathbf{v} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $\mathbf{w} = 4\mathbf{i}+5\mathbf{j}+6\mathbf{k}$ (ii) $\mathbf{v} = 2\mathbf{i}-\mathbf{j}+6\mathbf{k}$, $\mathbf{w} = -\mathbf{i}+\mathbf{j}-3\mathbf{k}$

9. (suitable for group discussion) Decide whether the following statements are true for all 2×2 matrices A and B:

(i) $\det(AB) = (\det A)(\det B)$ (ii) $\det(A+B) = (\det A) + (\det B)$

(iii) $\det(2A) = 2\det A$ (iv) $\det(-A) = \det A$

10.* Use the multiplicative property of the determinant to verify that if A is an invertible matrix then $\det A \neq 0$ and $\det A^{-1} = (\det A)^{-1}$.

Further Exercises:

11. Explain briefly why a square matrix with two identical rows or two identical columns has zero determinant.

12. Make sense of the expression $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$ where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are geometric vectors. Calculate it using the formula $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ in each of the following cases:

(i) $\mathbf{u} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{w} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

(ii) $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$, $\mathbf{w} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$

13. Determine the values of λ for which $\det(A - \lambda I) = 0$ in each case:

(i) $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ (iii)* $A = \begin{bmatrix} -3 & 0 & 2 \\ -4 & -1 & 4 \\ -4 & -4 & 7 \end{bmatrix}$

14.* Explain how the formula $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ follows from the formula

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} .$$

15.* Verify that if A is a 2×2 or a 3×3 matrix then $\det A = \det A^T$ where A^T is the transpose of A, obtained by interchanging rows and columns ('reflecting in the diagonal').

Short Answers:

1. $1, \ 1, \ -1, \ 2, \ -2, \ 3$

2. $$\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = 2\begin{vmatrix} 3 & 4 \\ -2 & 8 \end{vmatrix} + 3\begin{vmatrix} -1 & 4 \\ -7 & 8 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ -7 & -2 \end{vmatrix} = 64 + 60 - 46 = 78$$

3. (i) $$\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = \begin{vmatrix} -3 & -2 \\ -2 & 8 \end{vmatrix} + 3\begin{vmatrix} 2 & -2 \\ -7 & 8 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ -7 & -2 \end{vmatrix} = -28 + 6 + 100 = 78$$

(ii) $$\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = -7\begin{vmatrix} -3 & -2 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ -1 & 4 \end{vmatrix} + 8\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 42 + 12 + 24 = 78$$

(iii) $$\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = -2\begin{vmatrix} -1 & 3 \\ -7 & -2 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ -7 & -2 \end{vmatrix} + 8\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = -46 + 100 + 24 = 78$$

4. (i) -16 (ii) 0 (iii) 1 (iv) -1 (v) 1 (vi) 0 (vii) -96

5. (i) 10 (ii) -36 (iii) 0 (iv) 2 (v) -90 (vi) 16

6. Apply $R_2 \to R_2 + 2R_1$, followed by $R_3 \to R_3 + 4R_1$, followed by $C_2 \to C_2 + C_1$, followed by expansion down the third column and evaluation of 2×2 determinant.

7. (i) -14 (ii) 0 (iii) 32 **8.** (i) $-3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ (ii) $-3\mathbf{i} + \mathbf{k}$

9. (i) true (ii) false (iii) false (iv) true

10. Both follow quickly from the observation that $(\det A^{-1})(\det A) = \det(A^{-1}A) = \det I = 1$.

11. By subtracting one row or column from its identical row or column, a matrix is obtained with a row or column of zeros, which has zero determinant.

12. $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ (i) -5 (ii) 21 **13.** (i) $2, -3$ (ii) $2, 3$ (iii) $1, -1, 3$

14. $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot (w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

15. When A is 2×2 the answer is immediate. When A is 3×3 expand along the first rows of A and A^T and manipulate the expressions to see that they are equal.