Summer School **Excercise Sheet 9 Solutions** 2012

The short solutions suffice except for the following questions.

4. (i) $\begin{vmatrix} 5 & 2 \\ 3 & -2 \end{vmatrix} = 5(-2) - 2(3) = -16$ (ii) $\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix} = 6(1) - 2(3) = 0$

(iii) $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 0 - (-1) = 1$ (iv) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1$

(v) $\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} = -1 - (-2) = 1$

(vi) $\begin{vmatrix} 2 & 4 & 6 \\ 7 & 11 & 6 \\ -6 & -6 & 12 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 6 \\ 5 & 7 & 0 \\ -10 & -14 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 6 \\ 5 & 7 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 - 0 + 0 = 0$

(vii) $\begin{vmatrix} -4 & 3 & 3 \\ 8 & 7 & 3 \\ 4 & 3 & 3 \end{vmatrix} = \begin{vmatrix} -4 & 3 & 3 \\ 12 & 4 & 0 \\ 8 & 0 & 0 \end{vmatrix} = 8\begin{vmatrix} 3 & 3 \\ 4 & 0 \end{vmatrix} = 8(0 - 12) = -96$

5. (i) $\begin{vmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -5 & -1 \end{vmatrix} = 5(-2)(-1) = 10$ (ii) $\begin{vmatrix} 3 & 3 & 8 \\ 0 & -6 & -7 \\ 0 & 0 & 2 \end{vmatrix} = 3(-6)(2) = -36$

(iii) $\begin{vmatrix} -4 & -5 & 11 \\ 0 & 0 & 0 \\ 2 & -1 & 2 \end{vmatrix} = -0 + 0 - 0 = 0$ (iv) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{vmatrix} = 1(-1)(-2) = 2$

(v) $\begin{vmatrix} 0 & 0 & 5 \\ 6 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = 5(6)(-3) = -90$ (vi) $\begin{vmatrix} 4 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ -6 & -3 & -7 & -1 \end{vmatrix} = 4(-2)(2)(-1) = 16$

7. (i) $\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 4 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 3 & 5 \\ 4 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix} = -9 - 5 = -14$

(ii) $\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 0$

(iii) $\begin{vmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 & 4 \\ 3 & -2 & -5 & 1 \\ 4 & -4 & -7 & 7 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 4 \\ -2 & -5 & 1 \\ -4 & -7 & 7 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & 9 \\ 0 & 1 & 23 \end{vmatrix}$

$$= -\begin{vmatrix} -1 & 9 \\ 1 & 23 \end{vmatrix} = -(-23 - 9) = 32$$

8. (i) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \mathbf{k} = -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$

(ii) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 6 \\ -1 & 1 & -3 \end{vmatrix} = \begin{vmatrix} -1 & 6 \\ 1 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 6 \\ -1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} \mathbf{k} = -3\mathbf{i} + \mathbf{k}$

9. (i) This is the usual multiplicative property, which is always true.

(ii) This is false. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\det(A+B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 = 0+0 = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = (\det A) + (\det B) .$$

(iii) This is false. In fact, always,

$$\det(2A) = \det(2IA) = \det(2I) \det A = 4 \det A \neq 2 \det A ,$$

except when $\det A = 0$.

(iv) This is true always since

$$\det(-A) = \det(-IA) = \det(-I) \det A = (-1)(-1) \det A = \det A .$$

10.* Suppose A is an invertible matrix. Then $AA^{-1} = I$, so, by the multiplicative property,

$$1 = \det I = \det(AA^{-1}) = (\det A)(\det A^{-1}) .$$

If $\det A = 0$ then $1 = 0(\det A^{-1}) = 0$ which is impossible. Hence $\det A \neq 0$. Dividing through gives

$$\det A^{-1} = \frac{1}{\det A} .$$

12. (i) $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \begin{vmatrix} 1 & -3 & 1 \\ 2 & 3 & -3 \\ -1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 1 \\ 0 & 9 & -5 \\ 0 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 9 & -5 \\ -1 & 0 \end{vmatrix} = -5$

(ii) $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \begin{vmatrix} 2 & -1 & -2 \\ 1 & 5 & 6 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -3 & -2 \\ 7 & 11 & 6 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ 7 & 11 \end{vmatrix} = 21$

13. (i) $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & -3-\lambda \end{vmatrix} = (2-\lambda)(-3-\lambda) = 0$ if and only if $\lambda = 2$ or -3.

(ii) $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)+2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0$ if and only if $\lambda = 2$ or 3.

(iii)* $\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 4 \\ -4 & -4 & 7-\lambda \end{vmatrix} = \begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 4 \\ 0 & \lambda-3 & 3-\lambda \end{vmatrix}$

$$= \begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 3-\lambda \\ 0 & \lambda-3 & 0 \end{vmatrix} = -(\lambda-3)\begin{vmatrix} -3-\lambda & 2 \\ -4 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\big((-3-\lambda)(3-\lambda)+8\big) = (3-\lambda)(\lambda^2-1) = (3-\lambda)(\lambda-1)(\lambda+1) = 0$$

if and only if $\lambda = 3$, 1 or -1.

14.* Put $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Then

$$\mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot (w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k})$$

$$= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} \right) \cdot (w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k})$$

$$= w_1\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} .$$

15.* If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = ad - cb = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \det A^T .$$

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ then $A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix}$ and

$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ek - fh) - b(dk - fg) + c(dh - eg)$$

$$= aek - afh - bdk + bfg + cdh - ceg$$

$$= aek - ahf - dbk + dhc + gbf - gec$$

$$= a(ek - hf) - d(bk - hc) + g(bf - ec)$$

$$= a\begin{vmatrix} e & h \\ f & k \end{vmatrix} - d\begin{vmatrix} b & h \\ c & k \end{vmatrix} + g\begin{vmatrix} b & e \\ c & f \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & k \end{vmatrix} = \det A^T .$$