THE UNIVERSITY OF SYDNEY

MATH1002 LINEAR ALGEBRA

Summer School	**Exercise Sheet 10**	2012

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Let M be a square matrix, $\mathbf{x}$ a nonzero column vector and λ a scalar such that

$$M\mathbf{x} = \lambda\mathbf{x} .$$

Then λ is called an *eigenvalue* of M and $\mathbf{x}$ is called an *eigenvector* of M associated with the eigenvalue λ.

(ii) The *eigenspace* of M associated with an eigenvalue λ is the collection

$$\left\{ \mathbf{v} \;\middle|\; M\mathbf{v} = \lambda\mathbf{v} \right\} = \left\{ \mathbf{v} \;\middle|\; (M - \lambda I)\mathbf{v} = \mathbf{0} \right\}$$

comprising all the eigenvectors of M associated with λ and the zero vector (which is never an eigenvector).

(iii) A scalar λ is an eigenvalue of a square matrix M if and only if

$$\det(M - \lambda I) = 0 .$$

(iv) The expression $\det(M - \lambda I)$ is always a polynomial in λ and is called the *characteristic polynomial* of M. Thus the eigenvalues of a matrix are precisely the roots of its characteristic polynomial.

(v) Finding the eigenspace corresponding to the eigenvalue λ of a matrix M is equivalent to solving the homogeneous system with coefficient matrix $M - \lambda I$. After the eigenspace has been found, substituting particular values of the parameters yields particular eigenvectors.

(vi) The eigenvalues of a triangular matrix are simply the diagonal entries.

Preparatory Exercises:

1. Find $A\mathbf{v}$ and $A\mathbf{w}$ where

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

By inspection, write down the two eigenvalues of A.

2. Find $B\mathbf{v}_1$, $B\mathbf{v}_2$ and $B\mathbf{v}_3$ where

$$B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

By inspection, write down the three eigenvalues of B.

3. Find the characteristic polynomial $\det(M - \lambda I)$ and its roots in each case:

(i) $M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (ii) $M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (iii) $M = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

Tutorial Exercises:

4. Factorise the determinant $\begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix}$, which is a quadratic in λ, find its roots and relate them to the matrix A from the first exercise.

5. Factorise the determinant $\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$, which is a cubic in λ, find its roots, and relate them to the matrix B from the second exercise.

6. Using answers from the third exercise, write down the eigenvalues of M in each case, and then find the corresponding eigenspaces:

(i) $M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (ii) $M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (iii) $M = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

7. Write down the eigenvalues immediately for the following triangular matrices, and then find all of the corresponding eigenspaces.

(i) $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (ii) $M = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$ (iii) $M = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

8. Find all eigenvalues and eigenspaces for the following matrices:

(i) $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ (ii)* $B = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ (iii)* $C = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$

9. (suitable for group discussion) Verify that if A is invertible and λ is an eigenvalue of A, then $\lambda \neq 0$ and λ^{-1} is an eigenvalue of A^{-1}. What can be said about eigenvalues of A^k where k is any integer?

10.* Use the multiplicative property of the determinant to verify that if A and B are square matrices of the same size, and B is invertible, then A and $B^{-1}AB$ have the same eigenvalues.

Further Exercises:

11. Find the eigenvalues and corresponding eigenvectors for $M = \begin{bmatrix} -3 & 0 & 2 \\ -4 & -1 & 4 \\ -4 & -4 & 7 \end{bmatrix}$.

12.* Suppose that $0 \le \theta \le \pi$. Verify that $M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has real eigenvalues if and only if $\theta = 0$ or π .

13.* Verify that a square matrix A has the same eigenvalues as its transpose A^T.

14.* Let A be a square matrix with eigenvalue λ . Prove the following implications:

(i) $A^2 = 0 \implies \lambda = 0$

(ii) $A^2 = A \implies \lambda = 0$ or $\lambda = 1$

(iii) $A^2 = I \implies \lambda = 1$ or $\lambda = -1$

15.* Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Verify that the characteristic polynomial of A is

$$\lambda^2 - (a+d)\lambda + ad - bc \; .$$

Now also verify that

$$A^2 - (a+d)A + (ad-bc)I = 0 \; .$$

This result says that, in matrix arithmetic, A is a root of its own characteristic polynomial, a special instance of the celebrated *Cayley-Hamilton Theorem.*

Short Answers:

1. $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$, 5 , -3 **2.** $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$, 0 , 1 , 3

3. (i) $(\lambda-1)(\lambda-2)$, 1 , 2 (ii) $(\lambda-1)(\lambda+1)$, 1 , -1 (iii) $(\lambda+3)(\lambda-2)$, -3 , 2

4. $(\lambda-5)(\lambda+3)$; roots 5 and -3 are eigenvalues of A

5. $\lambda(\lambda-1)(3-\lambda)$; roots 0 , 1 and 3 are eigenvalues of B

6. (i) eigenvalues 1, 2 with respective eigenspaces $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

(ii) eigenvalues 1, -1 with respective eigenspaces $\left\{ \begin{bmatrix} -t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

(iii) eigenvalues 2, -3 with respective eigenspaces $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} -3t \\ 2t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

7. (i) eigenvalue 1 with eigenspace $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

(ii) eigenvalues 2, -1 with eigenspaces $\left\{ \begin{bmatrix} -3t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

(iii) eigenvalues 3, 5 with eigenspaces $\left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} 3t \\ 2t \\ 4t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

8. (i) eigenvalue 0 with eigenspace $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

(ii) eigenvalues 4, -2 with eigenspaces $\left\{ \begin{bmatrix} t \\ t \\ 2t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} \middle| \, s, t \in \mathbb{R} \right\}$

(iii) eigenvalues 4, -2 with eigenspaces $\left\{ \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

9. Suppose $\mathbf{v}$ is an eigenvector for invertible A corresponding to λ. If $\lambda = 0$ then $\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\lambda\mathbf{v} = A^{-1}0\mathbf{v} = \mathbf{0}$, a contradiction. If k is any integer, $A^k\mathbf{v} = \lambda^k\mathbf{v}$.

10. $\det(B^{-1}AB - \lambda I) = \det(B^{-1}(A - \lambda I)B) = \det B^{-1} \det(A - \lambda I) \det B = \det(A - \lambda I)$

11. $3, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, 1, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, -1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

12. eigenvalues are $\pm \operatorname{cis} \theta$, which are real if and only if $\theta = 0$ or π.

13. $\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$

14. Let $\mathbf{v}$ be an eigenvector of A corresponding to λ.

(i) If $A^2 = 0$ and $\lambda \neq 0$ then $\mathbf{v} = \lambda^{-2}\lambda^2\mathbf{v} = \lambda^{-2}A^2\mathbf{v} = \lambda^{-2}0\mathbf{v} = \mathbf{0}$, a contradiction.

(ii) If $A^2 = A$ and $\lambda \neq 0$ then $\mathbf{v} = \lambda^{-1}\lambda\mathbf{v} = \lambda^{-1}A\mathbf{v} = \lambda^{-2}A^2\mathbf{v} = \lambda^{-1}\lambda^2\mathbf{v} = \lambda\mathbf{v}$, so that $(1-\lambda)\mathbf{v} = \mathbf{0}$, yielding $1 - \lambda = 0$, so that $\lambda = 1$.

(iii) If $A^2 = I$ then $\mathbf{v} = A^2\mathbf{v} = \lambda^2\mathbf{v}$, so that $(1-\lambda^2)\mathbf{v} = \mathbf{0}$, yielding $1 - \lambda^2 = 0$, so that $\lambda = 1$ or -1.

15. straightforward evaluations using the definitions