**1.** Observe that

$$A\mathbf{v} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\mathbf{v}$$

and

$$A\mathbf{w} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3\mathbf{w}$$

so that $\mathbf{v}$ is an eigenvector corresponding to eigenvalue $5$, and $\mathbf{w}$ an eigenvector corresponding to eigenvalue $-3$.

**2.** Observe that

$$B\mathbf{v}_1 = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0\mathbf{v}_1 ,$$

$$B\mathbf{v}_2 = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1\mathbf{v}_2 ,$$

$$B\mathbf{v}_3 = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 3\mathbf{v}_3 ,$$

so that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are eigenvectors corresponding to eigenvalues $0$, $1$, $3$ respectively.

**3.** (i) $\begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)$ with roots $\lambda = 1,\ 2$.

(ii) $\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$ with roots $\lambda = 1,\ -1$.

(iii) $\begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$ with roots $\lambda = -3,\ 2$.

**4.** $\begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 16 = \lambda^2 - 2\lambda - 15 = (\lambda-5)(\lambda+3)$ with roots $\lambda = 5$ and $-3$, which correspond to the eigenvalues of $A$ in the first exercise.

**5.** $\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & -1 \\ \lambda-1 & 3-\lambda & 1 \\ 0 & 1-\lambda & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 3-\lambda & 0 \\ 0 & 1-\lambda & -\lambda \end{vmatrix}$

$= (1-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 1-\lambda & -\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)(-\lambda) = \lambda(\lambda-1)(3-\lambda)$ with roots $\lambda = 0,\ 1$ and $3$, which correspond to the eigenvalues of $B$ in the second exercise.

**6.** (i) The roots of the characteristic polynomial are $\lambda = 1$, $2$ and these are the eigenvalues. To find the eigenspace corresponding to $\lambda = 1$:

$$M - \lambda I = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

corresponding to the system with one equation $y = 0$ with solution $x = t$, $y = 0$, yielding the eigenspace

$$\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to $\lambda = 2$:

$$M - \lambda I = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

with solution $x = 0$, $y = t$, yielding the eigenspace $\left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

(ii) The roots of the characteristic polynomial are $\lambda = 1$ and $-1$. To find the eigenspace corresponding to $\lambda = 1$:

$$M - \lambda I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

corresponding to the system $x + y = 0$ with solution $x = -t$, $y = t$, yielding the eigenspace

$$\left\{ \begin{bmatrix} -t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to $\lambda = -1$:

$$M - \lambda I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

with solution $x = t$, $y = t$, yielding the eigenspace $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

(iii) The roots of the characteristic polynomial are $\lambda = 2$ and $-3$. To find the eigenspace corresponding to $\lambda = 2$:

$$M - \lambda I = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

with solution $x = t$, $y = t$, yielding the eigenspace $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$. To find the eigenspace corresponding to $\lambda = -3$:

$$M - \lambda I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$$

with solution $x = -3t$, $y = 2t$, yielding the eigenspace $\left\{ \begin{bmatrix} -3t \\ 2t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

**7.** (i) The only eigenvalue is $\lambda = 1$. To find its corresponding eigenspace:

$$M - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

with solution $x = t$, $y = 0$, yielding the eigenspace $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

(ii) The eigenvalues are $\lambda = 2$, $-1$. To find the eigenspace corresponding to $\lambda = 2$:

$$M - \lambda I = \begin{bmatrix} 0 & 0 \\ -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

with solution $x = -3t$, $y = t$, yielding the eigenspace $\left\{ \begin{bmatrix} -3t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.
To find the eigenspace corresponding to $\lambda = -1$:

$$M - \lambda I = \begin{bmatrix} 3 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

with solution $x = 0$, $y = t$, yielding the eigenspace $\left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

(iii) The eigenvalues are $\lambda = 3$, $5$. To find the eigenspace corresponding to $\lambda = 3$:

$$M - \lambda I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

with solution $x = t$, $y = 0$, $z = 0$, yielding the eigenspace $\left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

To find the eigenspace corresponding to $\lambda = 5$:

$$M - \lambda I = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

with solution $x = 3t$, $y = 2t$, $z = 4t$ and eigenspace $\left\{ \begin{bmatrix} 3t \\ 2t \\ 4t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

**8.** (i) $\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) + 1 = \lambda^2$, so the only eigenvalue is $\lambda = 0$.

To find its corresponding eigenspace: $A - \lambda I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ with solution $x = t$, $y = t$ and eigenspace $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

(ii)* $$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 0 & 3 \\ -2-\lambda & -2-\lambda & 3 \\ 0 & -2-\lambda & 4-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 0 & 3 \\ 0 & -2-\lambda & 0 \\ 0 & -2-\lambda & 4-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(-2-\lambda)(4-\lambda) = (\lambda+2)^2(4-\lambda) \text{ with eigenvalues } \lambda = 4 \text{ and } -2.$$

To find the eigenspace corresponding to $\lambda = 4$:

$$B - \lambda I = \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix},$$

with solution $x = t$, $y = t$, $z = 2t$ and eigenspace $\left\{ \begin{bmatrix} t \\ t \\ 2t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

To find the eigenspace corresponding to $\lambda = -2$:

$$B - \lambda I = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with solution $x = s - t$, $y = s$, $z = t$ and eigenspace $\left\{ \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} \middle| \, s, t \in \mathbb{R} \right\}$.

(iii)* $$\begin{vmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 0 & -1 \\ -2-\lambda & 4-\lambda & -1 \\ 0 & 4-\lambda & -2-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 0 & -1 \\ 0 & 4-\lambda & 0 \\ 0 & 4-\lambda & -2-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(4-\lambda)(-2-\lambda) = (\lambda+2)^2(4-\lambda) \text{ with eigenvalues } \lambda = 4 \text{ and } -2.$$

To find the eigenspace corresponding to $\lambda = 4$:

$$C - \lambda I = \begin{bmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ -7 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

with solution $x = 0$, $y = t$, $z = t$ and eigenspace $\left\{ \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

To find the eigenspace corresponding to $\lambda = -2$:

$$C - \lambda I = \begin{bmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

with solution $x = t$, $y = t$, $z = 0$ and eigenspace $\left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

**9.** Suppose that $\mathbf{v}$ is an eigenvector for an invertible matrix $A$ corresponding to the eigenvalue $\lambda$. If $\lambda = 0$ then

$$\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\lambda\mathbf{v} = A^{-1}0\mathbf{v} = \mathbf{0},$$

which contradicts that $\mathbf{v}$ is nonzero. Hence $\lambda \neq 0$. From $A\mathbf{v} = \lambda\mathbf{v}$ we deduce

$$A^{-1}\mathbf{v} = A^{-1}\lambda^{-1}\lambda\mathbf{v} = \lambda^{-1}A^{-1}A\mathbf{v} = \lambda^{-1}I\mathbf{v} = \lambda^{-1}\mathbf{v},$$

so that $\mathbf{v}$ is an eigenvector of $A^{-1}$ corresponding to the eigenvalue $\lambda^{-1}$. If $k$ is any positive integer then

$$A^k\mathbf{v} = A^{k-1}A\mathbf{v} = A^{k-1}\lambda\mathbf{v} = \lambda A^{k-1}\mathbf{v} = \lambda^2 A^{k-2}\mathbf{v} = \ldots = \lambda^k\mathbf{v},$$

so that $\mathbf{v}$ is an eigenvector of $A^k$ corresponding to the eigenvalue $\lambda^k$. If $k$ is any negative integer, then the same argument, using $A^{-1}$ in place of $A$, yields that $\mathbf{v}$ is an eigenvector of $A^k$ corresponding to the eigenvalue $\lambda^k$. Finally, if $k = 0$ then certainly $1 = \lambda^0$ is an eigenvalue of $I = A^0$ with eigenvector $\mathbf{v}$.

**10.*** By the multiplicative property of the determinant and distributivity,

$$\begin{aligned}\det(B^{-1}AB - \lambda I) &= \det(B^{-1}AB - \lambda B^{-1}B) = \det(B^{-1}AB - B^{-1}\lambda IB) \\ &= \det(B^{-1}(A - \lambda I)B) = \det B^{-1} \det(A - \lambda I) \det B \\ &= \det B^{-1} \det B \det(A - \lambda I) = \det(A - \lambda I).\end{aligned}$$

Since their characteristic polynomials are identical, the matrices $A$ and $B^{-1}AB$ have the same eigenvalues.

**11.** $$\begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 4 \\ -4 & -4 & 7-\lambda \end{vmatrix} = \begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 4 \\ 0 & \lambda-3 & 3-\lambda \end{vmatrix} = \begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 3-\lambda \\ 0 & \lambda-3 & 0 \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} -3-\lambda & 2 \\ -4 & 3-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 1) \text{ with eigenvalues } \lambda = 3,\ 1 \text{ and } -1.$$

To find an eigenvector corresponding to $\lambda = 3$:

$$M - \lambda I = \begin{bmatrix} -6 & 0 & 2 \\ -4 & -4 & 4 \\ -4 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 6 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix},$$

with solution $x = t$, $y = 2t$, $z = 3t$. Thus an eigenvector is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

To find an eigenvector corresponding to $\lambda = 1$:

$$M - \lambda I = \begin{bmatrix} -4 & 0 & 2 \\ -4 & -2 & 4 \\ -4 & -4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & -2 & 2 \\ 0 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

with solution $x = t$, $y = 2t$, $z = 2t$. Thus an eigenvector is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

To find an eigenvector corresponding to $\lambda = -1$:

$$M - \lambda I = \begin{bmatrix} -2 & 0 & 2 \\ -4 & 0 & 4 \\ -4 & -4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

with solution $x = t$, $y = t$, $z = t$. Thus an eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

**12.*** Observe that

$$\begin{aligned}\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix} &= (\cos\theta-\lambda)^2+\sin^2\theta \\ &= \cos^2\theta-2\lambda\cos\theta+\lambda^2+\sin^2\theta \\ &= \lambda^2-2\lambda\cos\theta+1\ ,\end{aligned}$$

with roots

$$\lambda \;=\; \frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2} \;=\; \cos\theta\pm i\sin\theta \;=\; \pm\,\text{cis}\ \theta\,.$$

Thus the eigenvalues are real if and only if cis $\theta$ is real, which occurs precisely when cis $\theta=\pm1$, that is, $\theta=0$ or $\pi$.

**13.*** Observe, by properties of transpose, that

$$\det(A-\lambda I) \;=\; \det(A-\lambda I)^T \;=\; \det(A^T-\lambda I^T) \;=\; \det(A^T-\lambda I)\ ,$$

so that $A$ and $A^T$ have identical characteristic polynomials and therefore the same eigenvalues.

**14.*** Let $\mathbf{v}$ be an eigenvector of $A$ corresponding to $\lambda$.

(i) Suppose $A^2=0$. If $\lambda\neq0$ then

$$\mathbf{v} \;=\; \lambda^{-2}\lambda^2\mathbf{v} \;=\; \lambda^{-2}A^2\mathbf{v} \;=\; \lambda^{-2}0\mathbf{v} \;=\; \mathbf{0}\,,$$

which contradicts that an eigenvector is nonzero. Hence $\lambda=0$.

(ii) Suppose $A^2=A$ and $\lambda\neq0$. Then

$$\mathbf{v} \;=\; \lambda^{-1}\lambda\mathbf{v} \;=\; \lambda^{-1}A\mathbf{v} \;=\; \lambda^{-2}A^2\mathbf{v} \;=\; \lambda^{-1}\lambda^2\mathbf{v} \;=\; \lambda\mathbf{v}\,,$$

so that $(1-\lambda)\mathbf{v}=\mathbf{0}$. But $\mathbf{v}\neq\mathbf{0}$, so $1-\lambda=0$, giving $\lambda=1$.

(iii) Suppose $A^2=I$. Then

$$\mathbf{v} \;=\; I\mathbf{v} \;=\; A^2\mathbf{v} \;=\; \lambda^2\mathbf{v}\,,$$

so that $(1-\lambda^2)\mathbf{v}=\mathbf{0}$. But $\mathbf{v}\neq\mathbf{0}$, so $1-\lambda^2=0$, giving $\lambda=1$ or $-1$.

**15.*** Observe that

$$\det(A-\lambda I) \;=\; \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \;=\; (a-\lambda)(d-\lambda)-bc \;=\; \lambda^2-(a+d)\lambda+ad-bc\ ,$$

and

$$\begin{aligned} &A^2-(a+d)A+(ad-bc)I \\ &= \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} - \begin{bmatrix} a^2+da & ab+db \\ ac+dc & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} a^2+bc-a^2-da+ad-bc & ab+bd-ab-db \\ ca+dc-ac-dc & cb+d^2-ad-d^2+ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \;=\; 0\ ,\end{aligned}$$

so that $A$ is a root of its characteristic polynomial.