Summer School **Exercise Sheet 11** 2012

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) A square matrix D is *diagonal* if all entries off the diagonal are zero. If D and E are diagonal then DE is also diagonal, and its diagonal entries are simply the products of corresponding diagonal entries of D and E. Thus the diagonal elements of D^n are just the nth powers of the diagonal elements of D.

(ii) Let M be a square $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \ldots \mathbf{v}_n$. Then

$$MP = PD$$

where D is the diagonal matrix with eigenvalues down the diagonal and P the matrix with corresponding eigenvectors as columns. If P is invertible then

$$M = PDP^{-1} \qquad \text{and} \qquad D = P^{-1}MP \,.$$

In this case we say that M is *diagonalisable*.

(iii) In the preceding discussion, if the eigenvalues are all different then P is invertible and M is diagonalisable.

(iv) If M is diagonalisable then powers of M can be found easily by the formula

$$M^n = PD^nP^{-1} \,.$$

Preparatory Exercises:

1. Find the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} .$$

2. Write down an invertible matrix P and a diagonal matrix D such that

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} = PDP^{-1} \,.$$

What is P^{-1}? What is D^n where n is any positive integer?

3. Evaluate

$$A^n = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}^n = PD^nP^{-1}$$

for any positive integer n. Use your answer to find A^3 and A^4.

Tutorial Exercises:

4. The matrix $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ has eigenvalues 2 and 4 with corresponding eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

(i) Write down an invertible matrix P and a diagonal matrix D such that

$$B = PDP^{-1}.$$

(ii) Find a formula for B^n, and use it to find B^3 and B^4.

5. The matrix $C = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ has eigenvalues 0, 1 and 3 with corresponding eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ respectively.

(i) Write down an invertible matrix P and diagonal matrix D such that

$$C = PDP^{-1}.$$

(ii) Find a formula for C^n, and use it to find C^4.

6. Find eigenvalues and corresponding eigenvectors for $M = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

7. Write down an invertible matrix P and a diagonal matrix D such that

$$M = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = PDP^{-1}.$$

8.* Evaluate

$$M^n = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n = PD^nP^{-1}$$

for any positive integer n. Use your answer to find M^4.

9. (suitable for group discussion) Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for a matrix M corresponding to different eigenvalues λ_1 and λ_2. Explain why $\mathbf{v}_1$ cannot be a scalar multiple of $\mathbf{v}_2$.

10.* Three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are said to be *linearly independent* if

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = \mathbf{0} \quad \Longrightarrow \quad \alpha = \beta = \gamma = 0,$$

where α, β, γ are scalars. Explain why three eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ corresponding to three different eigenvalues λ_1, λ_2, λ_3 of a matrix M must be linearly independent.

Further Exercises:

11. Diagonalise $M = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and find M^n for any positive integer n.

12. Diagonalise $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ and find M^n for any positive integer n.

13.* Prove that $M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not diagonalisable.

14.* Consider the matrix $M = \begin{bmatrix} 1/2 & 2/5 \\ 1/2 & 3/5 \end{bmatrix}$, whose entries are positive and the columns add to 1. It is an example of a *regular stochastic* matrix. It is a theorem about regular stochastic matrices M that

$$\lim_{n\to\infty} M^n = \begin{bmatrix} \mathbf{v} & \mathbf{v} \end{bmatrix}$$

where $\mathbf{v}$ is the unique *steady state vector* of M, that is, $\mathbf{v}$ is the unique eigenvector corresponding to eigenvalue 1 whose entries add up to 1. Diagonalise M and verify this limiting behaviour in this particular example.

15.* The sequence of *Fibonacci numbers* is obtained by writing down 1 twice, and obtaining each successive number by adding the previous two numbers together:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

If we let x_n denote the nth Fibonacci number then

$$x_1 = x_2 = 1\,,\quad x_n = x_{n-1} + x_{n-2} \quad \text{for } n \geq 3\,,$$

so that

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Diagonalise $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ to find a general formula for the nth Fibonacci number.

Short Answers:

1. eigenvalues 2 and 3 with eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively

2. $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $D^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$

3. $A^n = \begin{bmatrix} 2^{n+1} - 3^n & -2^{n+1} + 2(3^n) \\ 2^n - 3^n & -2^n + 2(3^n) \end{bmatrix}$, $A^3 = \begin{bmatrix} -11 & 38 \\ -19 & 46 \end{bmatrix}$, $A^4 = \begin{bmatrix} -49 & 130 \\ -65 & 146 \end{bmatrix}$

4. $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 2^{n-1} + 2(4^{n-1}) & -2^{n-1} + 2(4^{n-1}) \\ -2^{n-1} + 2(4^{n-1}) & 2^{n-1} + 2(4^{n-1}) \end{bmatrix}$, $\begin{bmatrix} 36 & 28 \\ 28 & 36 \end{bmatrix}$, $\begin{bmatrix} 136 & 120 \\ 120 & 136 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1+3^{n-1} & 3^{n-1} & -1 \\ -1+2(3^{n-1}) & 2(3^{n-1}) & 1 \\ 3^{n-1} & 3^{n-1} & 0 \end{bmatrix}$ $\begin{bmatrix} 28 & 27 & -1 \\ 53 & 54 & 1 \\ 27 & 27 & 0 \end{bmatrix}$

6. $1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, 2, \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}, -1, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 5 & 1 \\ 1 & -3 & -3 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

8. $\frac{1}{6}\begin{bmatrix} -3+5(2^{n+1})-(-1)^n & -9+5(2^{n+1})-(-1)^n & -12+5(2^{n+1})+2(-1)^n \\ 3-6(2^n)+3(-1)^n & 9-6(2^n)+3(-1)^n & 12-6(2^n)-6(-1)^n \\ 2^{n+1}-2(-1)^n & 2^{n+1}-2(-1)^n & 2^{n+1}+4(-1)^n \end{bmatrix}$,

$\begin{bmatrix} 26 & 25 & 25 \\ -15 & -14 & -15 \\ 5 & 5 & 6 \end{bmatrix}$

9. Argue by contradiction. Suppose $\mathbf{v}_1 = \alpha\mathbf{v}_2$ and apply M to both sides.

10. Suppose $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = \mathbf{0}$, apply M twice and rearrange to deduce that one of the scalars is zero. Reduce to the previous exercise to deduce that the other scalars are zero.

11. $\begin{bmatrix} 2^n & 2^n-1 \\ 0 & 1 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & 2^n-1 & 2^n-1 \\ 0 & 2^n & 2^n-3^n \\ 0 & 0 & 3^n \end{bmatrix}$

13. Suppose $P^{-1}MP$ is diagonal where $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Deduce that $ad-bc=0$, contradicting that P is invertible.

14. $\mathbf{v} = \begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix}$, $M^n = \frac{1}{9}\begin{bmatrix} 4+5(1/10)^n & 4-4(1/10)^n \\ 5-5(1/10)^n & 5+4(1/10)^n \end{bmatrix} \to \begin{bmatrix} \mathbf{v} & \mathbf{v} \end{bmatrix}$

15. eigenvalues of M are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$,

$$M^n = \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1^{n+1}-\lambda_2^{n+1} & \lambda_1\lambda_2^{n+1}-\lambda_2\lambda_1^{n+1} \\ \lambda_1^n-\lambda_2^n & \lambda_1\lambda_2^n-\lambda_2\lambda_1^n \end{bmatrix},$$

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$