Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Geometric definition of dot product: If \( \mathbf{v} \) and \( \mathbf{w} \) are vectors and \( \theta \) is the angle between them, then

\[
\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos \theta ,
\]

so that

\[
\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}.
\]

(ii) Algebraic definition of dot product: If \( \mathbf{v} = a \mathbf{i} + b \mathbf{j} + c \mathbf{k} \) and \( \mathbf{w} = d \mathbf{i} + e \mathbf{j} + f \mathbf{k} \) then

\[
\mathbf{v} \cdot \mathbf{w} = ad + be + cf.
\]

(iii) The angle between two vectors is zero or acute if their dot product is positive. The angle is obtuse or \( 180^\circ \) if the dot product is negative. Two vectors are mutually perpendicular if the dot product is zero.

(iv) Cauchy-Schwarz Inequality: \( |\mathbf{v} \cdot \mathbf{w}| \leq |\mathbf{v}||\mathbf{w}| \).

(v) Commutativity of dot product: \( \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \).

(vi) Distributivity of dot over plus: \( (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \).

(vii) If \( \mathbf{v} \) is any vector then \( \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \), so \( |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \).

(viii) If \( \mathbf{v} \) and \( \mathbf{w} \) are vectors and \( \lambda \) is a scalar then \( (\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda (\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (\lambda \mathbf{w}) \).

(ix) The vector projection of \( \mathbf{v} \) in the direction of \( \mathbf{w} \) is \( \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \mathbf{w} \), which is the best approximation of \( \mathbf{v} \) using a scalar multiple of \( \mathbf{w} \).

(x) The scalar component of \( \mathbf{v} \) in the direction of \( \mathbf{w} \) is \( \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|} \), which is plus or minus the magnitude of the vector projection (minus in the case that the angle is obtuse or \( 180^\circ \)).

(xi) The vector component of \( \mathbf{v} \) orthogonal to \( \mathbf{w} \) is the difference between \( \mathbf{v} \) and its vector projection, which is

\[
\mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \mathbf{w}.
\]
Preparatory Exercises (answers below):

1. Given that
   \[ u = 2i - j + k, \quad v = i - 2j + 2k, \quad w = 3i - k, \]
   find
   (i) \( u \cdot v \)  (ii) \( u \cdot w \)  (iii) \( v \cdot w \)  (iv) \( u \cdot u \)  (v) \( v \cdot v \)  (vi) \( w \cdot w \)
   (vii) \( |u| \)  (viii) \( |v| \)  (ix) \( |w| \)  (x) \( u \cdot (v + w) \)  (xi) \( u \cdot (v - w) \)

2. Let \( u, v, w \) be as in the previous exercise. Let \( \alpha \) be the angle between \( u \) and \( v \), \( \beta \) be the angle between \( u \) and \( w \), and \( \gamma \) the angle between \( v \) and \( w \). Find
   (i) \( \cos \alpha \)  (ii) \( \cos \beta \)  (iii) \( \cos \gamma \)

3. Given that
   \[ a = 2i - j + 2k, \quad b = i + j - k, \quad c = 3i + 6j, \]
   determine whether the following are true or false:
   (i) The angle between \( a \) and \( b \) is acute. (ii) The angle between \( b \) and \( c \) is acute.
   (iii) The vectors \( a \) and \( c \) are mutually perpendicular.
   (iv) The angle between the vectors \( a + b \) and \( b - c \) is obtuse.

Tutorial Exercises:

4. Given that \( P = (8, 4, -1) \), \( Q = (6, 3, -4) \) and \( R = (7, 5, -5) \), find
   \[ \overrightarrow{QP}, \quad |\overrightarrow{QP}|, \quad \overrightarrow{QR}, \quad |\overrightarrow{QR}|, \quad \overrightarrow{QP} \cdot \overrightarrow{QR}, \]
   and the cosine of \( \angle PQR \).

5. Given that \( u = i - 2j \) and \( v = -2i + j \), find
   (i) \( u \cdot v \)  (ii) \( \hat{u} \)  (iii) \( \hat{v} \)  (iv) \( \frac{u \cdot v}{|u|} \)  (v) \( \frac{u \cdot v}{|v|} \)  (vi) \( \frac{u \cdot v}{|u||v|} \)
   (vii) \( \frac{u \cdot v}{|u|^2} u \)  (viii) \( \frac{u \cdot v}{|v|^2} v \)  (ix) \( v - \frac{u \cdot v}{|u|^2} u \)  (x) \( u - \frac{u \cdot v}{|v|^2} v \)
   (xi) the cosine of the angle between \( u \) and \( v \)
   (xii) the scalar component of \( u \) in the direction of \( v \)
   (xiii) the scalar component of \( v \) in the direction of \( u \)
   (xiv) the vector projection of \( u \) in the direction of \( v \)
   (xv) the vector projection of \( v \) in the direction of \( u \)
   (xvi) the vector component of \( u \) orthogonal to \( v \)
   (xvii) the vector component of \( v \) orthogonal to \( u \)
6. Use the dot product to verify that if \( \mathbf{v} \) and \( \mathbf{w} \) are any vectors and \( \mathbf{w} \) is nonzero, then

\[
\mathbf{w} \quad \text{and} \quad \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \mathbf{w}
\]

are mutually perpendicular.

7. Given that \( \mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \) and \( \mathbf{v} = -4\mathbf{i} + 4\mathbf{j} - \mathbf{k} \), find

(i) the cosine of the angle between \( \mathbf{u} \) and \( \mathbf{v} \)
(ii) the scalar component of \( \mathbf{u} \) in the direction of \( \mathbf{v} \)
(iii) the scalar component of \( \mathbf{v} \) in the direction of \( \mathbf{u} \)
(iv) the vector projection of \( \mathbf{u} \) in the direction of \( \mathbf{v} \)
(v) the vector projection of \( \mathbf{v} \) in the direction of \( \mathbf{u} \)
(vi) the vector component of \( \mathbf{u} \) orthogonal to \( \mathbf{v} \)
(vii) the vector component of \( \mathbf{v} \) orthogonal to \( \mathbf{u} \)

8. Verify that if \( \mathbf{a} \) and \( \mathbf{b} \) are vectors of the same length then

\[
\mathbf{a} + \mathbf{b} \quad \text{and} \quad \mathbf{a} - \mathbf{b}
\]

are mutually perpendicular.

9. (suitable for group discussion) Use vectors to find the following angles in a cube:

(i) between a major diagonal (between opposite vertices) and an edge,
(ii) between a major diagonal and a face diagonal,
(iii) between diagonals on adjacent faces,
(iv) between major diagonals.

10. * Use vectors to show that any angle inscribed in a semicircle is a right angle.

Further Exercises:

11. Resolve the vector \( \mathbf{u} = 5\mathbf{i} + \mathbf{j} + 6\mathbf{k} \) into a sum of two vectors, one of which is parallel and the other perpendicular to \( \mathbf{v} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \).

12. (Homework) Find the (vector) components of the force \( 15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k} \) newtons in the direction of and orthogonal to

\[
\begin{align*}
\text{(i)} & \quad -\mathbf{i} + \mathbf{j} & \quad \text{(ii)} & \quad 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}
\end{align*}
\]
13. Prove that if $a$ and $b$ are mutually perpendicular vectors then
\[ |a + b|^2 = |a|^2 + |b|^2. \]
Interpret this result in terms of a well-known fact about triangles.

14.* (Homework) Verify that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.

15.* Prove that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus (that is, has all sides of equal length).

Answers to Preparatory Exercises:

1. (i) 6 (ii) 5 (iii) 1 (iv) 6 (v) 9 (vi) 10 (vii) \( \sqrt{6} \) (viii) 3
   (ix) \( \sqrt{10} \) (x) 11 (xi) 1

2. (i) \( \frac{\sqrt{6}}{3} \) (ii) \( \frac{\sqrt{15}}{6} \) (iii) \( \frac{\sqrt{10}}{30} \)

3. (i) false (ii) true (iii) true (iv) true