Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

**Important Ideas and Useful Facts:**

(i) A line in space is determined by two points, or by one point and a direction.

(ii) A plane in space is determined either by three non-collinear points, or by one point and a perpendicular (normal) direction.

(iii) If the vector \( \mathbf{v} \) points in the direction of a line \( \mathcal{L} \) containing the point \( P_0 \), then the **parametric vector equation** of \( \mathcal{L} \) is

\[
\mathbf{r} - \mathbf{r}_0 = t\mathbf{v} \quad \text{or equivalently} \quad \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}
\]

where \( \mathbf{r} \) is the position vector of a typical point on \( \mathcal{L} \), \( \mathbf{r}_0 \) is the position vector of \( P_0 \) and \( t \) is a parameter which varies over all real numbers.

(iv) If the vector \( \mathbf{v} = ai + bj + ck \) points in the direction of a line \( \mathcal{L} \) containing the point \( P_0(x_0, y_0, z_0) \), then the **parametric scalar equations** of \( \mathcal{L} \) are

\[
\begin{align*}
x &= x_0 + ta \\
y &= y_0 + tb \\
z &= z_0 + tc
\end{align*}
\]

and the **Cartesian equations** are (in the case that \( a, b, c \) are all nonzero):

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.
\]

(v) If the vector \( \mathbf{n} \) is normal to a plane \( \mathcal{P} \) containing the point \( P_0 \), then the **vector equation** of \( \mathcal{P} \) is

\[
(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0 \quad \text{or equivalently} \quad \mathbf{r} \cdot \mathbf{n} = \mathbf{r}_0 \cdot \mathbf{n}
\]

where \( \mathbf{r} \) is the position vector of a typical point and \( \mathbf{r}_0 \) is the position vector of \( P_0 \).

(vi) If the vector \( \mathbf{n} = ai + bj + ck \) is normal to the plane \( \mathcal{P} \) containing the point \( P_0(x_0, y_0, z_0) \), then the **Cartesian equation** of \( \mathcal{P} \) is

\[
ax + by + cz = d
\]

where \( d = ax_0 + by_0 + cz_0 \).

(vii) If \( P_1, P_2, P_3 \) are non-collinear points on a plane, then a normal vector to the plane is

\[
\mathbf{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}.
\]
Preparatory Exercises:

1. Find parametric vector, parametric scalar and Cartesian equations of the line passing through the point \((2, 3, 5)\) in the direction of \(\mathbf{i} + 3\mathbf{j} - \mathbf{k}\).

2. Find vector and Cartesian equations of the plane containing the point \((2, 3, 5)\) with normal vector \(\mathbf{i} + 3\mathbf{j} - \mathbf{k}\).

3. Let \(P = (1, 2, 3), Q = (-1, -2, -3)\) and \(R = (4, -4, 4)\).
   (i) Express \(\overrightarrow{PQ}\) and \(\overrightarrow{PR}\) in Cartesian form.
   (ii) Find the cross product \(\overrightarrow{PQ} \times \overrightarrow{PR}\).
   (iii) Find the Cartesian equation of the plane containing \(P, Q, R\).

Tutorial Exercises:

4. Find parametric vector, parametric scalar and Cartesian equations of the line passing through \(P\) in the direction of \(\mathbf{v}\) in each of the following cases:
   (i) \(P = (1, 0, -1), \quad \mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}\)
   (ii) \(P = (-6, 5, 2), \quad \mathbf{v} = 2\mathbf{i} - 5\mathbf{k}\)
   (iii) \(P = (0, 1, -1), \quad \mathbf{v} = \mathbf{i} + 2\mathbf{j}\)
   (iv) \(P = (2, 3, -3), \quad \mathbf{v} = -\mathbf{i}\)

5. Find parametric vector, parametric scalar and Cartesian equations of the line passing through \(P\) and \(Q\) in each of the following cases:
   (i) \(P = (-4, 3, 5), \quad Q = (-2, 4, -1)\)
   (ii) \(P = (0, 5, 0), \quad Q = (5, 0, -5)\)

6. Find vector and Cartesian equations of the plane containing \(P\) having normal vector \(\mathbf{n}\) in each of the following cases:
   (i) \(P = (4, -1, 0), \quad \mathbf{n} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}\)
   (ii) \(P = (7, 5, -3), \quad \mathbf{n} = 2\mathbf{i} + \mathbf{k}\)
   (iii) \(P = (0, 0, -9), \quad \mathbf{n} = \mathbf{i} + \mathbf{j} - \mathbf{k}\)
   (iv) \(P = (-6, 5, 6), \quad \mathbf{n} = \mathbf{j}\)

7. Find a Cartesian equation for the plane containing
   \[ P = (6, 7, -2), \quad Q = (0, -8, 11), \quad R = (14, -3, 9). \]

8. (Homework) The planes \(x + y + z = 2\) and \(x - y + 3z = 0\) intersect in a line \(\mathcal{L}\).
   (i) Find a point on \(\mathcal{L}\).
   (ii) Use cross products to find a vector pointing in the direction of \(\mathcal{L}\).
   (iii) Write down parametric scalar and Cartesian equations for \(\mathcal{L}\).
9. (suitable for group discussion) Two lines in space are skew if they are not parallel and do not intersect. The following lines are not parallel. Show that they are not skew, by finding their point of intersection:

\[ L_1: \mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 4\mathbf{k}), \quad L_2: \mathbf{r} = 6\mathbf{i} - 6\mathbf{j} + \mathbf{k} + s(-7\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}) \]

10.* Find the distance from \( P(3, 0, -1) \) to the plane \( P \) described by the equation

\[ 4x + 2y - z = 6 \]

Find the closest point to \( P \) which lies on \( P \).

Further Exercises:

11. For each of (i)–(vii), find two matching descriptions from (a)–(n).

(i) line containing \((0, 0, 0)\) in the direction of \(\mathbf{i} + \mathbf{j} + \mathbf{k}\)
(ii) line containing \((-1, 2, -1)\) in the direction of \(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\)
(iii) line containing \((-1, 2, -1)\) and \((0, 0, -2)\)
(iv) plane containing \((0, 0, 0)\) with normal vector \(\mathbf{i} + \mathbf{j} + \mathbf{k}\)
(v) plane containing \((-1, 2, -1)\) with normal vector \(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\)
(vi) plane containing \((-1, 2, -1)\), \((0, 0, -2)\) and \((1, 3, 3)\)
(vii) plane containing \((-1, 2, -1)\), \((0, 0, -2)\) and \((1, 3, 2)\)

(a) \(x + y + z = 0\) (b) \(x = y = z\) (c) \(x + y - z = 2\)
(d) \(x + 1 = \frac{y - 2}{-2} = \frac{z + 1}{-2}\) (e) \(7x + 6y - 5z = 10\)
(f) \(x + 1 = \frac{y - 2}{-2} = \frac{z + 1}{-1}\) (g) \(x - 2y - 2z = -3\)
(h) \((\mathbf{r} + 2\mathbf{k}) \cdot (7\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}) = 0\) (i) \(\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 5\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})\)
(j) \((\mathbf{r} + 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0\) (k) \(\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0\)
(l) \((\mathbf{r} + 3\mathbf{i}) \cdot (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = 0\) (m) \(\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - \mathbf{k})\)
(n) \(\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} - t(\mathbf{i} + \mathbf{j} + \mathbf{k})\)

12. Find Cartesian equations of the line passing through \((1, 0, -2)\) and perpendicular to the plane \(3x - 4y + z = 6\).

13. Verify that the line

\[ \frac{x - 3}{2} = \frac{y - 4}{3} = \frac{z - 5}{4} \]

is parallel to the plane \(4x + 4y - 5z = 14\).
14. Find the Cartesian equation of the plane containing \((1,1,1)\) and the line 

\[
\frac{x - 4}{-2} = \frac{y + 3}{3} = \frac{z - 1}{3}.
\]

15. *(Homework)* Find the distance from \(P(2,1,1)\) to the line \(L\) given by the equations

\[
x - 1 = \frac{y - 1}{3} = \frac{z + 4}{-1}.
\]

Find the closest point to \(P\) lying on \(L\).

Selected Short Answers:

1. \(\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + t(\mathbf{i} + 3\mathbf{j} - \mathbf{k}), \quad \begin{aligned} x &= 2 + t \\ y &= 3 + 3t \\ z &= 5 - t \end{aligned} \quad t \in \mathbb{R}, \quad x - 2 = \frac{y - 3}{3} = \frac{z - 5}{-1}

2. \(\mathbf{r} \cdot (\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 6, \quad x + 3y - z = 6\).

3. (i) \(-2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}, \quad 3\mathbf{i} - 6\mathbf{j} + \mathbf{k}\)  
   (ii) \(-40\mathbf{i} - 16\mathbf{j} + 24\mathbf{k}\)  
   (iii) \(5x + 2y - 3z = 0\)

4. (i) \(\frac{x - 1}{2} = \frac{y}{2} = \frac{z + 1}{-1}\)  
   (ii) \(\frac{x + 6}{2} = \frac{z - 2}{-5}, \quad y = 5\)  
   (iii) \(x = \frac{y - 1}{2}, \quad z = -1\)  
   (iv) \(y = 3, \quad z = -3\)

5. (i) \(\frac{x + 4}{2} = y - 3 = \frac{z - 5}{-6}\)  
   (ii) \(x = 5 - y = -z\)

6. (i) \(3x + y - 4z = 11\)  
   (ii) \(2x + z = 11\)  
   (iii) \(x + y - z = 9\)  
   (iv) \(y = 5\)

7. \(-7x + 34y + 36z = 124\)

9. Point of intersection is \((-8, 4, -11)\).  

10. \(\sqrt{21}/3, \quad (5/3, -2/3, -2/3)\)

11. (i) (b)(a)  
   (ii) (d)(i)  
   (iii) (f)(m)  
   (iv) (a)(k)  
   (v) (g)(l)  
   (vi) (e)(h)  
   (vii) (c)(j)

12. \(\frac{x - 1}{3} = \frac{y}{-4} = z + 2\)

13. \((2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \cdot (4\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) = 0\)

14. \(12x + 9y + 5z = 26\)