The University of Sydney

MATH1002 Linear Algebra

Summer School **Excercise Sheet 11 Solutions** 2008

1. $\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)+2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$ with roots 2 and 3. But $A - 2I = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ so an eigenvector corresponding to the eigenvalue 2 is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $A - 3I = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so an eigenvector corresponding to the eigenvalue 3 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2. We form P by listing the eigenvectors as columns and D by listing the eigenvalues down the diagonal in the same order:

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then, using the formula for inverting a 2×2 matrix, $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Raising each diagonal entry to its nth power yields $D^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$.

3. We have

$$\begin{aligned} A^n &= PD^nP^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2^{n+1} & 3^n \\ 2^n & 3^n \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2^{n+1} - 3^n & -2^{n+1} + 2(3^n) \\ 2^n - 3^n & -2^n + 2(3^n) \end{bmatrix}, \end{aligned}$$

so in particular $A^3 = \begin{bmatrix} -11 & 38 \\ -19 & 46 \end{bmatrix}$ and $A^4 = \begin{bmatrix} -49 & 130 \\ -65 & 146 \end{bmatrix}$.

4. (i) We may take $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

(ii) We have

$$\begin{aligned} B^n &= PD^nP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 4^n \end{bmatrix}\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} -2^n & 4^n \\ 2^n & 4^n \end{bmatrix}\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 2^{n-1} + 2(4^{n-1}) & -2^{n-1} + 2(4^{n-1}) \\ -2^{n-1} + 2(4^{n-1}) & 2^{n-1} + 2(4^{n-1}) \end{bmatrix}, \end{aligned}$$

so in particular $B^3 = \begin{bmatrix} 36 & 28 \\ 28 & 36 \end{bmatrix}$ and $B^4 = \begin{bmatrix} 136 & 120 \\ 120 & 136 \end{bmatrix}$.

5. (i) We may take $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(ii) By row reducing an augmented matrix we discover

$$P^{-1} = \frac{1}{3}\begin{bmatrix} -1 & -1 & 3 \\ 3 & 0 & -3 \\ 1 & 1 & 0 \end{bmatrix},$$

and so we have

$$\begin{aligned} C^n = PD^nP^{-1} &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{bmatrix}\begin{bmatrix} -1 & -1 & 3 \\ 3 & 0 & -3 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1+3^{n-1} & 3^{n-1} & -1 \\ -1+2(3^{n-1}) & 2(3^{n-1}) & 1 \\ 3^{n-1} & 3^{n-1} & 0 \end{bmatrix}, \end{aligned}$$

so in particular $C^4 = \begin{bmatrix} 28 & 27 & -1 \\ 53 & 54 & 1 \\ 27 & 27 & 0 \end{bmatrix}$.

6. $$\begin{vmatrix} 3-\lambda & 2 & 1 \\ -2 & -1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 1 \\ \lambda-1 & -3-\lambda & 1 \\ 0 & 1+2\lambda & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & -3-\lambda & 2 \\ 0 & 1+2\lambda & -\lambda \end{vmatrix}$$

$= (1-\lambda)\begin{vmatrix} -3-\lambda & 2 \\ 1+2\lambda & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-\lambda-2) = (1-\lambda)(\lambda-2)(\lambda+1)$ with roots 1, 2 and -1. But

$$M - I = \begin{bmatrix} 2 & 2 & 1 \\ -2 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so an eigenvector corresponding to the eigenvalue 1 is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$;

$$M - 2I = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

so an eigenvector corresponding to the eigenvalue 2 is $\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$; and

$$M + I = \begin{bmatrix} 4 & 2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

so an eigenvector corresponding to the eigenvalue -1 is $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$.

7. We may take $P = \begin{bmatrix} -1 & 5 & 1 \\ 1 & -3 & -3 \\ 0 & 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

8.* By row reducing an augmented matrix we discover

$$P^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 9 & 12 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix},$$

and so we have

$$\begin{aligned} M^n &= PD^nP^{-1} = \frac{1}{6}\begin{bmatrix} -1 & 5 & 1 \\ 1 & -3 & -3 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & (-1)^n \end{bmatrix}\begin{bmatrix} 3 & 9 & 12 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} -3+5(2^{n+1})-(-1)^n & -9+5(2^{n+1})-(-1)^n & -12+5(2^{n+1})+2(-1)^n \\ 3-6(2^n)+3(-1)^n & 9-6(2^n)+3(-1)^n & 12-6(2^n)-6(-1)^n \\ 2^{n+1}-2(-1)^n & 2^{n+1}-2(-1)^n & 2^{n+1}+4(-1)^n \end{bmatrix}, \end{aligned}$$

so in particular $M^4 = \begin{bmatrix} 26 & 25 & 25 \\ -15 & -14 & -15 \\ 5 & 5 & 6 \end{bmatrix}$.

9. We argue by contradiction. Suppose

$$\mathbf{v}_1 = \alpha\mathbf{v}_2$$

for some scalar α. Then $\alpha \neq 0$, since $\mathbf{v}_1$ is nonzero (being an eigenvector). Then

$$\lambda_2\mathbf{v}_2 = M\mathbf{v}_2 = M\alpha^{-1}\mathbf{v_1} = \alpha^{-1}\lambda_1\mathbf{v}_1 = \alpha^{-1}\lambda_1\alpha\mathbf{v_2} = \lambda_1\mathbf{v}_2 .$$

Hence

$$(\lambda_1 - \lambda_2)\mathbf{v}_2 = \mathbf{0},$$

which implies $\lambda_1 - \lambda_2 = 0$ (now since $\mathbf{v}_2$ is nonzero), contradicting that λ_1 and λ_2 are different. Hence $\mathbf{v}_1$ is not a scalar mutliple of $\mathbf{v}_2$.

10.* Suppose α, β and γ are scalars such that

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = \mathbf{0}.$$

Mutliplying through by λ_1 gives

$$\lambda_1\alpha\mathbf{v}_1 + \lambda_1\beta\mathbf{v}_2 + \lambda_1\gamma\mathbf{v}_3 = \mathbf{0}.$$

Multiplying through by M, using the definitions of eigenvectors and eigenvalues, gives

$$\alpha\lambda_1\mathbf{v}_1 + \beta\lambda_2\mathbf{v}_2 + \gamma\lambda_3\mathbf{v}_3 = \mathbf{0}.$$

Subtracting gives

$$(\lambda_1 - \lambda_2)\beta\mathbf{v}_2 + (\lambda_1 - \lambda_3)\gamma\mathbf{v}_3 = \mathbf{0}.$$

Mutliplying through by λ_2 gives

$$\lambda_2(\lambda_1 - \lambda_2)\beta\mathbf{v}_2 + \lambda_2(\lambda_1 - \lambda_3)\gamma\mathbf{v}_3 = \mathbf{0},$$

and by M gives

$$(\lambda_1-\lambda_2)\lambda_2\beta\mathbf{v}_2+(\lambda_1-\lambda_3)\lambda_3\gamma\mathbf{v}_3 \;=\; \mathbf{0}\,.$$

Again subtracting gives

$$(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\gamma\mathbf{v}_3 \;=\; \mathbf{0}\,.$$

But $\lambda_1-\lambda_3\neq 0$, $\lambda_2-\lambda_3\neq 0$ and $\mathbf{v}_3\neq\mathbf{0}$. Thus $\gamma=0$ and so

$$\alpha\mathbf{v}_1+\beta\mathbf{v}_2 = \;\mathbf{0}\,.$$

If $\alpha\neq 0$ or $\beta\neq 0$ then one of $\mathbf{v_1}$ or $\mathbf{v}_2$ is a scalar multiple of the other, contradicting the previous exercise. Hence

$$\alpha \;=\; \beta \;=\; \gamma \;=\; 0\,,$$

which proves that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

11. The matrix is triangular so the eigenvalues are the diagonal entries. Quickly one discovers that $\begin{bmatrix}1\\0\end{bmatrix}$ is an eigenvector corresponding to 2, and $\begin{bmatrix}-1\\1\end{bmatrix}$ an eigenvector corresponding to 1. Hence

$$\begin{aligned}M^n \;&=\; \begin{bmatrix}1&-1\\0&1\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}^n\begin{bmatrix}1&-1\\0&1\end{bmatrix}^{-1} \;=\; \begin{bmatrix}1&-1\\0&1\end{bmatrix}\begin{bmatrix}2^n&0\\0&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}\\ &=\; \begin{bmatrix}2^n&2^n-1\\0&1\end{bmatrix}\,.\end{aligned}$$

12. The matrix is triangular so the eigenvalues are the diagonal entries. Quickly one discovers that $\begin{bmatrix}1\\0\\0\end{bmatrix}$ corresponds to 1, $\begin{bmatrix}1\\1\\0\end{bmatrix}$ corresponds to 2 and $\begin{bmatrix}0\\1\\-1\end{bmatrix}$ corresponds to 3. Hence

$$\begin{aligned}M^n \;&=\; \begin{bmatrix}1&1&0\\0&1&1\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}^n\begin{bmatrix}1&1&0\\0&1&1\\0&0&-1\end{bmatrix}^{-1}\\ &=\; \begin{bmatrix}1&1&0\\0&1&1\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2^n&0\\0&0&3^n\end{bmatrix}\begin{bmatrix}1&-1&-1\\0&1&1\\0&0&-1\end{bmatrix}\\ &=\; \begin{bmatrix}1&2^n-1&2^n-1\\0&2^n&2^n-3^n\\0&0&3^n\end{bmatrix}\,.\end{aligned}$$

13.* We argue by contradiction. Suppose that M is diagonalisable, so $P^{-1}MP$ is diagonal for some invertible matrix $P=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. But

$$P^{-1}MP \;=\; \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}\begin{bmatrix}2&1\\0&2\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix} \;=\; \frac{1}{ad-bc}\begin{bmatrix}*&d^2\\-c^2&*\end{bmatrix}\,,$$

which implies $c=d=0$, so that $ad-bc=0$, contradicting that P is invertible.

14.* $\begin{vmatrix} 1/2-\lambda & 2/5 \\ 1/2 & 3/5-\lambda \end{vmatrix} = (\lambda-1/2)(\lambda-3/5)-1/5 = \lambda^2-11\lambda/10+1/10 = (\lambda-1)(\lambda-1/10)$ with roots 1 and $1/10$. But

$$M - I = \begin{bmatrix} -1/2 & 2/5 \\ 1/2 & -2/5 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/5 \\ 0 & 0 \end{bmatrix}$$

so the eigenspace corresponding to the eigenvalue 1 is $\left\{ \begin{bmatrix} 4t \\ 5t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$. The element of this eigenspace whose entries add to 1 is

$$\mathbf{v} = \begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix},$$

which is the unique steady state vector of M. Also

$$M - \frac{1}{10}I = \begin{bmatrix} 2/5 & 2/5 \\ 1/2 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

so an eigenvector corresponding to the eigenvalue $1/10$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence

$$\begin{aligned} M^n &= \begin{bmatrix} 4 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/10 \end{bmatrix}^n \begin{bmatrix} 4 & -1 \\ 5 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{9} \begin{bmatrix} 4 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1/10)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -5 & 4 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 4+5(1/10)^n & 4-4(1/10)^n \\ 5-5(1/10)^n & 5+4(1/10)^n \end{bmatrix}. \end{aligned}$$

But $(1/10)^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} M^n = \frac{1}{9} \begin{bmatrix} 4 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 4/9 & 4/9 \\ 5/9 & 5/9 \end{bmatrix} = \begin{bmatrix} \mathbf{v} & \mathbf{v} \end{bmatrix},$$

as the general theory predicted.

15.* $\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1$ with roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

But $M - \lambda_1 I \sim \begin{bmatrix} 1 & -\lambda_1 \\ 0 & 0 \end{bmatrix}$ so an eigenvector for λ_1 is $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$. Similarly an eigenvector for λ_2 is $\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$. Hence

$$\begin{aligned} M^n &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1\lambda_2^{n+1} - \lambda_2\lambda_1^{n+1} \\ \lambda_1^n - \lambda_2^n & \lambda_1\lambda_2^n - \lambda_2\lambda_1^n \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = M^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n-1}(1-\lambda_2) - \lambda_2^{n-1}(1-\lambda_1) \\ \lambda_1^{n-2}(1-\lambda_2) - \lambda_2^{n-2}(1-\lambda_1) \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^n - \lambda_2^n \\ \lambda_1^{n-1} - \lambda_2^{n-1} \end{bmatrix}$$

yielding finally the formula for the nth Fibonacci number:

$$x_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$