

4. We have

$$\overrightarrow{QP} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad |\overrightarrow{QP}| = \sqrt{14}, \quad \overrightarrow{QR} = \mathbf{i} + 2\mathbf{j} - \mathbf{k},$$

$$|\overrightarrow{QR}| = \sqrt{6}, \quad \overrightarrow{QP} \cdot \overrightarrow{QR} = 2(1) + 1(2) + 3(-1) = 1,$$

so that the cosine of $\angle PQR$ is $\frac{1}{\sqrt{6}\sqrt{14}} = \frac{1}{2\sqrt{21}}$.

5. (i) -4 (ii) $\frac{1}{\sqrt{5}}(\mathbf{i} - 2\mathbf{j})$ (iii) $\frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j})$ (iv) $-\frac{4}{\sqrt{5}}$ (v) $-\frac{4}{\sqrt{5}}$ (vi) $-\frac{4}{5}$
 (vii) $-\frac{4}{5}(\mathbf{i} - 2\mathbf{j})$ (viii) $\frac{4}{5}(2\mathbf{i} - \mathbf{j})$ (ix) $-\frac{3}{5}(2\mathbf{i} + \mathbf{j})$ (x) $-\frac{3}{5}(\mathbf{i} + 2\mathbf{j})$
 (xi) the cosine of the angle between \mathbf{u} and \mathbf{v} equals $-\frac{4}{5}$.
 (xii) the scalar component of \mathbf{u} in the direction of \mathbf{v} equals $-\frac{4}{\sqrt{5}}$.
 (xiii) the scalar component of \mathbf{v} in the direction of \mathbf{u} equals $-\frac{4}{\sqrt{5}}$.
 (xiv) the vector projection of \mathbf{u} in the direction of \mathbf{v} equals $\frac{4}{5}(2\mathbf{i} - \mathbf{j})$.
 (xv) the vector projection of \mathbf{v} in the direction of \mathbf{u} equals $-\frac{4}{5}(\mathbf{i} - 2\mathbf{j})$.
 (xvi) the vector component of \mathbf{u} orthogonal to \mathbf{v} equals $-\frac{3}{5}(\mathbf{i} + 2\mathbf{j})$.
 (xvii) the vector component of \mathbf{v} orthogonal to \mathbf{u} equals $-\frac{3}{5}(2\mathbf{i} + \mathbf{j})$.

6. The vectors are perpendicular because

$$\begin{aligned} \mathbf{w} \cdot \left(\mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \mathbf{w} \right) &= \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \mathbf{w} \right) = \mathbf{w} \cdot \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} (\mathbf{w} \cdot \mathbf{w}) \\ &= \mathbf{w} \cdot \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} |\mathbf{w}|^2 = \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} \\ &= \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} = 0. \end{aligned}$$

7. (i) the cosine of the angle between \mathbf{u} and \mathbf{v} equals $\frac{2}{3\sqrt{33}}$.
 (ii) the scalar component of \mathbf{u} in the direction of \mathbf{v} equals $\frac{2}{\sqrt{33}}$.
 (iii) the scalar component of \mathbf{v} in the direction of \mathbf{u} equals $\frac{2}{3}$.
 (iv) the vector projection of \mathbf{u} in the direction of \mathbf{v} equals $\frac{2}{33}(-4\mathbf{i} + 4\mathbf{j} - \mathbf{k})$.
 (v) the vector projection of \mathbf{v} in the direction of \mathbf{u} equals $\frac{2}{9}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$.
 (vi) the vector component of \mathbf{u} orthogonal to \mathbf{v} equals $\frac{1}{33}(41\mathbf{i} + 58\mathbf{j} + 68\mathbf{k})$.
 (vii) the vector component of \mathbf{v} orthogonal to \mathbf{u} equals $-\frac{1}{9}(38\mathbf{i} - 32\mathbf{j} + 13\mathbf{k})$.

8. Observe that

$$\begin{aligned}(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} - |\mathbf{b}|^2 \\ &= |\mathbf{a}|^2 - |\mathbf{b}|^2 = 0,\end{aligned}$$

so that $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are mutually perpendicular.

9. (i) The cosine of the angle between a major diagonal and an edge is

$$\frac{|\mathbf{i} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})|}{|\mathbf{i}| |\mathbf{i} + \mathbf{j} + \mathbf{k}|} = \frac{1}{\sqrt{3}},$$

yielding an angle of approximately 54 degrees.

(ii) The cosine of the angle between a major diagonal and a face diagonal is

$$\frac{|(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})|}{|\mathbf{i} + \mathbf{j}| |\mathbf{i} + \mathbf{j} + \mathbf{k}|} = \frac{2}{\sqrt{6}},$$

yielding an angle of approximately 35 degrees.

(iii) The cosine of the angle between diagonals on adjacent faces is

$$\frac{|(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + \mathbf{k})|}{|\mathbf{i} + \mathbf{j}| |\mathbf{i} + \mathbf{k}|} = \frac{1}{2},$$

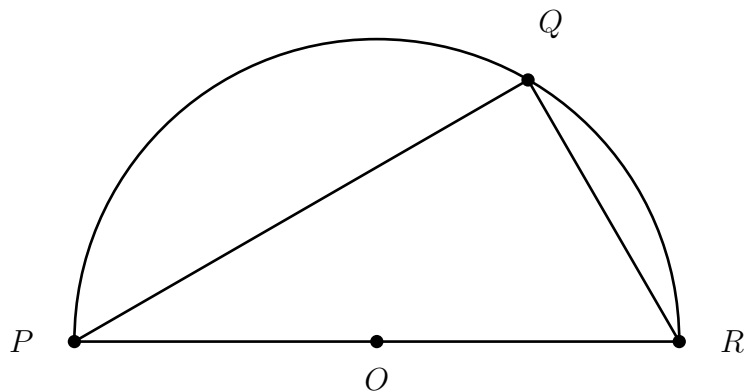
yielding an angle of exactly 60 degrees.

(iv) The cosine of the angle between major diagonals is

$$\frac{|(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + \mathbf{k})|}{|\mathbf{i} + \mathbf{j} + \mathbf{k}| |\mathbf{i} - \mathbf{j} + \mathbf{k}|} = \frac{1}{3},$$

yielding an angle of approximately 71 degrees.

10.* Consider a semicircle and points O , P , Q , R as shown:



To show the angle at Q is a right-angle, it suffices to show $\overrightarrow{QP} \cdot \overrightarrow{QR} = 0$. But this follows from Exercise 8, since

$$\overrightarrow{QP} = \overrightarrow{QO} + \overrightarrow{OP} \quad \text{and} \quad \overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR} = \overrightarrow{QO} - \overrightarrow{OP}$$

and the fact that $|\overrightarrow{QO}| = |\overrightarrow{OP}|$, the radius of the circle.

11. The projection of \mathbf{u} in the direction of \mathbf{v} is

$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \frac{15 - 6 + 12}{9 + 36 + 4} \mathbf{v} = \frac{3}{7}(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}).$$

The component of \mathbf{u} orthogonal to \mathbf{v} is

$$\mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \mathbf{u} - \frac{3}{7}(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) = \frac{1}{7}(26\mathbf{i} + 25\mathbf{j} + 36\mathbf{k}).$$

Thus, the decomposition of \mathbf{u} as a sum of vectors, the first parallel to \mathbf{v} and the second perpendicular to \mathbf{v} , is

$$\mathbf{u} = \frac{3}{7}(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) + \frac{1}{7}(26\mathbf{i} + 25\mathbf{j} + 36\mathbf{k}).$$

12. (i) The component of the force in the direction of $-\mathbf{i} + \mathbf{j}$ is

$$\frac{(15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j})}{|-\mathbf{i} + \mathbf{j}|^2} (-\mathbf{i} + \mathbf{j}) = \frac{5}{2}(-\mathbf{i} + \mathbf{j}) \text{ newtons,}$$

and the component orthogonal to $-\mathbf{i} + \mathbf{j}$ is

$$15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k} - \frac{5}{2}(-\mathbf{i} + \mathbf{j}) = \frac{1}{2}(35\mathbf{i} + 35\mathbf{j} + 12\mathbf{k}) \text{ newtons.}$$

- (ii) The component of the force in the direction of $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is

$$\frac{(15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + \mathbf{k})}{|2\mathbf{i} - 3\mathbf{j} + \mathbf{k}|^2} (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = -\frac{12}{7}(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \text{ newtons,}$$

and the component orthogonal to $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is

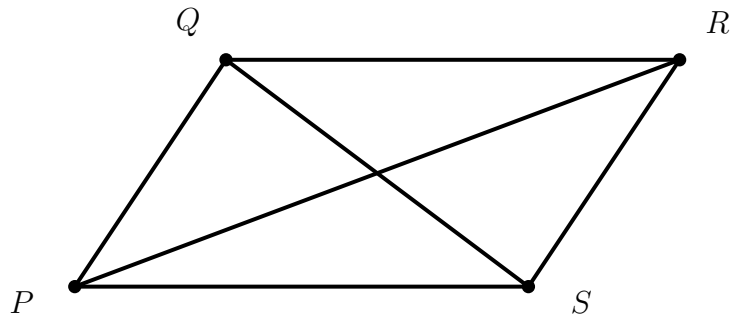
$$15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k} + \frac{12}{7}(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = \frac{1}{7}(129\mathbf{i} + 104\mathbf{j} + 54\mathbf{k}) \text{ newtons.}$$

13. If \mathbf{a} and \mathbf{b} are mutually perpendicular then $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = 0$, so that

$$\begin{aligned} |\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 + 0 + 0 + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2. \end{aligned}$$

This is just the usual Theorem of Pythagoras where \mathbf{a} and \mathbf{b} label directed edges of a right-angled triangle.

- 14.* Consider the following parallelogram, and put $\mathbf{v} = \overrightarrow{PQ}$ and $\mathbf{w} = \overrightarrow{QR}$:



Then the sum of the squares of the lengths of the diagonals is

$$\begin{aligned}
 |\overrightarrow{PR}|^2 + |\overrightarrow{QS}|^2 &= \overrightarrow{PR} \cdot \overrightarrow{PR} + \overrightarrow{QS} \cdot \overrightarrow{QS} \\
 &= (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) + (\mathbf{w} - \mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}) \\
 &= \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} - 2\mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{v} \\
 &= 2\mathbf{v} \cdot \mathbf{v} + 2\mathbf{w} \cdot \mathbf{w} = 2(|\mathbf{v}|^2 + |\mathbf{w}|^2),
 \end{aligned}$$

which is the sum of the squares of the lengths of the sides.

- 15.* The diagonals of the parallelogram of the previous exercise are perpendicular if and only if

$$\overrightarrow{QS} \cdot \overrightarrow{PR} = 0,$$

that is,

$$(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} = |\mathbf{v}|^2 - |\mathbf{w}|^2 = 0,$$

that is,

$$|\mathbf{v}|^2 = |\mathbf{w}|^2,$$

that is,

$$|\mathbf{v}| = |\mathbf{w}|,$$

that is, the parallelogram is a rhombus.