Assumed Knowledge: Factorisation of expressions. Simple techniques of integration.

Objectives:

(7a) To be able to recognise a differential equation as a separable equation.

(7b) To be able to solve a separable equation by separation of variables.

(7c) To be able to perform integrations using trigonometric substitutions.

Exercises:

1. Find the general solutions of

(a) \( \frac{dy}{dx} = 1 + y^2 \)  
(b) \( \frac{dy}{dx} = y \cos x \)  
(c) \( (1 + x) \frac{dy}{dx} + y^2 = 0 \)

2. Evaluate \( \int \frac{x^2}{(x^2 + 1)^{1/2}} \, dx \) by making the substitution \( x = \sinh t \).

Hence solve the differential equation \( \frac{dy}{dx} = \frac{x^2 y}{(x^2 + 1)^{1/2}} \).

3. (a) Find the general solution of \( \frac{dy}{dx} = \cosec y \sqrt{x^2 + 4} \).

(b) Find the particular solution of \( \frac{dx}{dt} = \frac{x^2}{\cos^2 t} \) if \( x = 1 \) when \( t = 0 \).

4. (a) Given \( y = A \sqrt{x^2 + 1} \), where \( A \) is an arbitrary constant, show by substitution that it satisfies the differential equation \( \frac{dy}{dx} = \frac{xy}{x^2 + 1} \).

(b) Since the general solution of a first-order differential equation depends on one arbitrary constant, we see that the solution given in part (a) is the general solution of \( \frac{dy}{dx} = \frac{xy}{x^2 + 1} \). Now find the particular solution satisfying the initial condition \( y(0) = 1 \).

(c) Sketch the family of solution curves given in part (a), indicating the behaviour of solutions for \( A > 0 \), \( A = 0 \) and \( A < 0 \). Indicate also on your sketch the particular solution found in part (b).
Brief answers to selected exercises:

1.  (a)  \( y = \tan(x - C) \)
    (b)  \( y = Ae^{\sin x} \)

2.  \( \ln |y| = \frac{1}{2} x \sqrt{1 + x^2} - \frac{1}{2} \sinh^{-1} x + C \)

3.  (a)  \( -\cos y = \theta + C = \sinh^{-1} \frac{x}{2} + C \)
    (b)  \( x = \frac{1}{1 - \tan \theta} \)