Assumed Knowledge: Finding the roots of quadratic equations. Euler’s formula $e^{i\theta} = \cos \theta + i\sin \theta$.

Objectives:

(11a) To be able to write down the auxiliary (or characteristic) equation associated with a second-order differential equation with constant coefficients.

(11b) To be able to construct the solutions to such differential equations in terms of exponential and trigonometric functions.

Exercises:

1. Find the general solution of $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 4y = 0$.

2. Find the particular solution of $\frac{d^2y}{dt^2} - 9y = 0$ which satisfies the initial conditions $y = 3$ and $\frac{dy}{dt} = 3$ when $t = 0$.

3. Find the general solutions of these second-order homogeneous equations. In each case give your answer in terms of real functions.
   
   (a) $2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 0$.
   
   (b) $\frac{d^2y}{dx^2} + 3y = 0$.
   
   (c) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

4. Find the general solution of each of these second-order linear homogeneous equations. Hence find the particular solution for the given conditions. In each case give your answer in terms of real functions.

   (a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 20y = 0, \quad y(0) = y'(0) = 1$.
   
   (b) $\frac{d^2y}{dx^2} + 9y = 0, \quad y(0) = 1, \quad y(\pi/6) = 3$.
   
   (c) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = 0, \quad y(0) = 0, \quad \frac{dy}{dt} = 3$ when $t = 0$.
   
   (d) $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0, \quad x(0) = 1, \quad x(1) = 3e^2$. 
Brief answers to selected exercises:

1. \( e^{-t}(Ae^{\sqrt{5}t} + Be^{-\sqrt{5}t}) \)

2. \( y = 2e^{3t} + e^{-3t} \)

3. (a) \( y = Ae^{x/2} + Be^{3x} \)
   (b) \( A\cos \sqrt{3}x + B\sin \sqrt{3}x \)
   (c) \( y = e^x(A\cos x + B\sin x) \)

4. (a) \( y = \frac{1}{3}e^{-5x} + \frac{2}{3}e^{4x} \)
   (b) \( y = \cos 3x + 3\sin 3x \)
   (c) \( y = \sqrt{3}e^{-2t}\sin \sqrt{3}t \)
   (d) \( x = e^{2t} + 2te^{2t} \)