Assumed Knowledge: Proportionality and inverse proportionality. Integration techniques. Taylor series and binomial series.

Objectives:

(6a) Given a verbal description of a simple model, to be able to express it as a mathematical equation.

(6b) To be able to recognise an ordinary differential equation.

(6c) To be able to sketch the solution curves for a first-order differential equation from its direction field.

Exercises:

1. Find the general solution by antidifferentiation and sketch the solution curves of:
   
   (a) \( \frac{dy}{dx} = \cos 2x \)
   
   **Solution:** \( y = \int \cos 2x \, dx = \frac{1}{2} \sin 2x + C \). The set of solution curves is a set of sine curves, each with amplitude \( \frac{1}{2} \) and period \( \pi \), displaced vertically one above the other.

   (b) \( \frac{dy}{dx} = \cosh x \)
   
   **Solution:** \( y = \int \cosh x \, dx = \sinh x + C \). The set of solution curves is obtained by displacing \( \sinh x \) vertically.

2. Construct a mathematical model of the following experiment (given below) by:

   (i) determining and naming suitable independent and dependent variables;

   (ii) constructing a differential equation;

   (iii) relating the other conditions to the variables you have introduced.

   Heat tends to flow from hot bodies to cold bodies. Newton observed that the rate at which temperature rises or falls within a body is proportional to the temperature difference between the body and the surrounding air. In the experiment to test Newton’s law, the surrounding air remained at a constant temperature and the body temperature was initially twice the temperature of the surrounding air.
Solution: The independent variable is \( t \) measuring time while the dependent variable is \( T \) measuring the temperature in the body at time \( t \). So \( T = T(t) \). Let \( T_0 \) be the temperature of the surrounding air. The equation is then \( dT/dt = -k(T - T_0) \) for some positive constant \( k \). Note the negative sign; this is to remind us that when the body is hotter than the surroundings, the temperature should be falling. The initial condition is given: \( T = 2T_0 \) when \( t = 0 \).

3. (a) Show that \( y = A(e^x - x) \), where \( A \) is an arbitrary constant, satisfies the differential equation

\[
\frac{dy}{dx} = \frac{y(e^x - 1)}{e^x - x}.
\]

Solution: Differentiate \( y = A(e^x - x) \) to get \( \frac{dy}{dx} = A(e^x - 1) \).

When we substitute \( y \) into the RHS we find \( \frac{y(e^x - 1)}{e^x - x} = A(e^x - 1) = \text{LHS} \), and so the given function satisfies the equation.

(b) Since the general solution of a first order-differential equation involves only one arbitrary constant, we see that the solution given in part (a) is the general solution of the differential equation. Find the particular solution satisfying the initial condition \( y(0) = -3 \).

Solution: We require that \( y(0) = -3 \), so we must have \( -3 = A(e^0 - 0) = A \), and the particular solution is \( y = -3(e^x - x) \).

(c) Sketch the solution curves given in part (a) in the special cases \( A = 1 \), \( A = 0 \) and \( A = -1 \).

Solution: Consider first the case \( A = 1 \), \( y = (e^x - x) \). Note that \( y(0) = 1 \).

Also, \( \frac{dy}{dx} = (e^x - 1) \) and so \( \frac{dy}{dx} = 0 \) when \( x = 0 \). Then since \( \frac{dy}{dx} < 0 \) for \( x < 0 \) and \( \frac{dy}{dx} > 0 \) for \( x > 0 \), we see that \( x = 0 \) is a global minimum. Also note that \( \frac{d^2y}{dx^2} = e^x > 0 \) for all \( x \). When \( A = -1 \), \( y = -e^x + x \) and the curve is simply a reflection in the \( x \)-axis of the curve \( y = e^x - x \). Finally, in the case \( A = 0 \) we obtain the particular solution \( y = 0 \).
4. Find the general solution of the following differential equations, and in each case give also the particular solution satisfying the initial condition $y(0) = 2$.

(a) $\frac{dy}{dx} = 20xe^{5x^2}$

**Solution:** General solution is $y = 2e^{5x^2} + C$. The condition $y(0) = 2$ then requires $2 = 2e^{5\times0^2} + C = 2 + C$, so $C = 0$, and the particular solution is $y = 2e^{5x^2}$.

(b) $\frac{dy}{dx} = 6x^2 + \cos x$

**Solution:** General solution is $y = 2x^3 + \sin x + C$. The condition $y(0) = 2$ then requires $2 = C$, so the particular solution is $y = 2x^3 + \sin x + 2$.

(c) $\frac{dy}{dx} = x \cos x$

**Solution:** Use integration by parts to get the general solution

$$y = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$ 

The initial condition $y(0) = 2$ then requires $2 = 1 + C$, so $C = 1$, and the particular solution is $y = x \sin x + \cos x + 1$.

(d) $\frac{dy}{dx} = x^2 e^x$

**Solution:** Using integration by parts (twice) we get the general solution

$$y = x^2 e^x - \int 2xe^x \, dx$$

$$= x^2 e^x - \left( 2xe^x - \int 2e^x \, dx \right)$$

$$= x^2 e^x - 2xe^x + 2e^x + C.$$ 

The initial condition $y(0) = 2$ then requires $2 = 2 + C$, so $C = 0$, and the particular solution is $y = x^2 e^x - 2xe^x + 2e^x$.

5. Newton’s law of gravitation states that the acceleration of an object at a distance $r$ from the centre of an object of mass $M$ is given by

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2},$$

where $G$ is the universal gravitational constant.

(a) Use the identity

$$\frac{d^2r}{dt^2} = \frac{d}{dr} \left( \frac{1}{2} \frac{d}{dt} v^2 \right),$$

where $v = dr/dt$, to show by integration that

$$v^2 - u^2 = 2GM \frac{1}{r} - \frac{2GM}{R},$$

when $v = u$ at $r = R$. 

Solution: Substituting,
\[ \frac{d}{dr} \left( \frac{1}{2} v^2 \right) = -\frac{GM}{r^2}, \]
and performing the antidifferentiation,
\[ \frac{1}{2} v^2 = - \int \frac{GM}{r^2} dr = \frac{GM}{r} + C. \]
We must choose \( C \) so that \( v = u \) when \( r = R \); so substituting, we want
\[ \frac{1}{2} u^2 = \frac{GM}{R} + C. \]
Replacing \( C \) by this expression gives (after rearrangement)
\[ v^2 - u^2 = \frac{2GM}{r} - \frac{2GM}{R}. \]
(b) Now write \( r = R + s \) where \( s \) is the height of the object above the surface of the Earth, radius \( R \) and mass \( M \). Use the binomial series to expand the factor \( (1 + s/R)^{-1} \) to show that, close to the surface of the Earth,
\[ v^2 \approx u^2 - 2gs, \]
for some constant \( g \).
Find the expression for \( g \).
Solution: Writing \( r = R + s \),
\[ v^2 - u^2 = \frac{2GM}{(R + s)} - \frac{2GM}{R} = \frac{2GM}{R} \left( 1 + \frac{s}{R} \right)^{-1} - \frac{2GM}{R}. \]
Using the binomial theorem on \( (1 + s/R)^{-1} \), we obtain
\[ v^2 = u^2 + \frac{2GM}{R} \left( 1 - \frac{s}{R} + \frac{s^2}{R^2} + \ldots \right) \]
\[ - \frac{2GM}{R} = u^2 - \frac{2GM}{R^2} s + \ldots \]
Retaining only the leading term,
\[ v^2 \approx u^2 - 2 \left( \frac{GM}{R^2} \right) s, \]
so \( g = GM/R^2 \).