Assumed Knowledge: Integration techniques.

Objectives:

(9a) To be able to distinguish between separable and linear first-order differential equations.

(9b) To be able to solve a linear equations utilising an integrating factor.

Exercises:

1. For each of these equations, determine whether it is separable or a first-order linear equation. If an equation is linear, write it in standard form \( \frac{dy}{dx} + p(x)y = q(x) \) (with a suitable renaming of variables where necessary) and identify the functions \( p \) and \( q \).

   (a) \( t \frac{dx}{dt} + x = \cos t \)

   **Solution:** This equation is linear.

   In standard form this is \( \frac{dx}{dt} + \frac{x}{t} = \frac{\cos t}{t} \) so \( p(t) = \frac{1}{t} \) and \( q(t) = \frac{\cos t}{t} \).

   (b) \( \frac{dy}{dx} = \frac{2x\sqrt{y}}{\sqrt{1+x^2}} \)

   **Solution:** This equation is separable.

2. Find the general solution of \( t \frac{dx}{dt} + x = \cos t \).

   **Solution:** Working with the standard form (see Question 1 part (a)), the integrating factor is \( r(t) = e^{\int(1/t)dt} = e^{\ln t} = t \). Multiply the standard form by the integrating factor:

   \[ t \frac{dx}{dt} + x = \cos t \]

   \[ t \frac{d}{dt}(tx) = \cos t \] .

   Integrate both sides with respect to \( t \).

   \[ tx = \sin t + C \] .

   So \( x = \frac{\sin t}{t} + \frac{C}{t} \).
3. (a) For each of the following differential equations, find the general solution and also the particular solution satisfying $y(1) = 0$.

(i) $\frac{dy}{dx} + 4y = e^{-2x}$

\textbf{Solution:} The integrating factor is $e^{\int 4 \, dx} = e^{4x}$, and multiplying our equation by this gives

$$\frac{d}{dx} \left( e^{4x} y \right) = e^{2x}$$

and thus

$$e^{4x} y = \frac{1}{2} e^{2x} + C,$$

which then gives the general solution

$$y = \frac{1}{2} e^{-2x} + Ce^{-4x}.$$

The condition $y(1) = 0$ gives $0 = \frac{1}{2} e^{-2} + Ce^{-4}$, and so $C = -\frac{1}{2} e^2$.

Thus the required particular solution is

$$y = \frac{1}{2} e^{-2x} \left( 1 - e^{2(1-x)} \right).$$

(ii) $\frac{dy}{dx} + (\sinh x)y = (2x)e^{-\cosh x}$

\textbf{Solution:} The integrating factor is $e^{\int \sinh x \, dx} = e^{\cosh x}$. Multiplying our equation by this integrating factor gives

$$\frac{d}{dx} \left( e^{\cosh x} y \right) = 2x,$$

and so

$$e^{\cosh x} y = x^2 + C,$$

which then gives the general solution

$$y = (x^2 + C)e^{-\cosh x}.$$

The condition $y(1) = 0$ means $0 = (1 + C)e^{(-\cosh(1))}$, and so $C = -1$.

Thus the required particular solution is

$$y = (x^2 - 1)e^{-\cosh x}.$$

(b) Find the general solution of the differential equation

$$\frac{dz}{dx} + (\cot x)z = -2x,$$

where we assume $0 < x < \pi$. 

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**Solution:** This is a first-order linear equation. Its integrating factor is

\[ e^\int \cot x \, dx = e^\int \frac{\cos x}{\sin x} \, dx = e^{\ln x} = x. \]

Multiplying our equation by this integrating factor gives

\[ (\sin x) \frac{dz}{dx} + (\cos x) z = -2x \sin x \]

that is

\[ \frac{d}{dx}(z \sin x) = -2x \sin x. \]

We then integrate by parts to get

\[ z \sin x = 2x \cos x - \int 2 \cos x \, dx = 2x \cos x - 2 \sin x + C, \]

and thus we obtain the general solution as

\[ z = \frac{2x \cos x - 2 \sin x + C}{\sin x}. \]

4. Consider the equation

\[ \frac{dy}{dx} + y \cos x = \cos x. \]

Solve this equation as a linear equation and then solve it as a separable equation. Are the solutions the same?

**Solution:** The equation is already in standard first-order linear form with \( p(x) = \cos x \). So the integrating factor is \( \exp \left( \int \cos x \, dx \right) = e^{\sin x} \). Thus

\[ \frac{d}{dx}(ye^{\sin x}) = \cos x \, e^{\sin x} , \]

\[ ye^{\sin x} = e^{\sin x} + C \]

\[ y = 1 + Ce^{-\sin x}. \]

Solving by separating the variables, we have

\[ \int \frac{1}{1 - y} \, dy = \int \cos x \, dx \]

\[ - \ln |1 - y| = \sin x + C \]

\[ \ln |1 - y| = - \sin x - C \]

\[ |1 - y| = e^{-C} e^{-\sin x} \]

\[ 1 - y = Ae^{-\sin x} \]

\[ y = 1 - Ae^{-\sin x}. \]

The way that the constants of integration occur in the two solution methods is slightly different but the solutions are, of course, the same. To see this, simply replace \( A \) with \(-C\) in the solution above.
5. In electronic circuit theory, circuits with a resistor and an inductance coil in series with a voltage applied across these two components are known as RL circuits. This is because the resistance of the resistor is conventionally given as $R$ ohms and the inductance of the coil is conventionally given as $L$ henries. The equation for the rate of change of the electric current $I$ in such a circuit is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{V}{L}$$

where $V$ is the voltage applied to the circuit. In a circuit with an applied AC current, $V$ will vary with time as $V = A\sin \omega t$. So, if $R$ and $L$ are constant the equation becomes

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{A\sin \omega t}{L}.$$ 

Solve this equation to find the general solution for $I$ as a function of $t$. Find the particular solution if the circuit has no current in it when it is switched on. What happens to the current as $t \to \infty$? How does the initial condition affect this long-term behaviour?

[Hint: $\int e^{au} \sin bu \, du = \frac{e^{au}}{a^2 + b^2} (a \sin bu - b \cos bu) + C$]

**Solution:** The equation is first-order linear with $p(t) = R/L$. So the integrating factor is

$$r(t) = \exp \left( \int \frac{R}{L} \, dt \right) = e^{Rt/L}.$$

So the differential equation becomes

$$\frac{d}{dt} (e^{Rt/L}I) = \frac{A\sin \omega t}{L} e^{Rt/L},$$

since $R$ and $L$ are constants. Integrating both sides with respect to $t$:

$$e^{Rt/L}I = \frac{A e^{Rt/L}}{L (R/L)^2 + \omega^2} \left( \frac{R}{L} \sin \omega t - \omega \cos \omega t \right) + C$$

$$I = \frac{A}{L (R/L)^2 + \omega^2} \left( \frac{R}{L} \sin \omega t - \omega \cos \omega t \right) + Ce^{-Rt/L}.$$

This is the general solution. To find the particular solution let $I = 0$ when $t = 0$. Then the equation gives

$$\frac{A}{L (R/L)^2 + \omega^2} \frac{-\omega}{\omega} + C = 0$$

so

$$C = \frac{A}{L (R/L)^2 + \omega^2} \omega$$

and the particular solution is

$$I = A \frac{1}{L (R/L)^2 + \omega^2} \left( \frac{R}{L} \sin \omega t - \omega \cos \omega t + \omega e^{-Rt/L} \right).$$

As $t \to \infty$, $e^{-Rt/L} \to 0$ leaving just the sine and cosine terms. So eventually the current will be a periodic function of time (that is it will oscillate) with a period of $2\pi/\omega$. The initial condition only contributed to the $e^{-Rt/L}$ term (the transient term). Hence it has no effect on the long term behaviour.