Assumed Knowledge: Integration techniques.

Objectives:

(10a) To be able to solve differential equations that are separable, linear or both.

(10b) To be able to construct and solve equations describing flow and mixing problems.

Exercises:

1. Classify and solve the differential equation \((1 + x^2) \frac{dy}{dx} = y - 2xy\).

**Solution:** This equation is both separable and linear.

As a separable equation:

\[
(1 + x^2) \frac{dy}{dx} = y - 2xy = y(1 - 2x)
\]

\[
\frac{dy}{dx} = \frac{1 - 2x}{1 + x^2} y
\]

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1 - 2x}{1 + x^2}
\]

\[
\int \frac{1}{y} \frac{dy}{dx} = \int \frac{1 - 2x}{1 + x^2} dx = \int \left( \frac{1}{1 + x^2} - \frac{2x}{1 + x^2} \right) dx
\]

\[
\ln |y| = \tan^{-1} x - \ln(1 + x^2) + C
\]

\[
y = A \exp \left( \tan^{-1} x + \ln \left( \frac{1}{1 + x^2} \right) \right) = \frac{A e^{\tan^{-1} x}}{1 + x^2}.
\]

In the standard form for first-order linear, the equation is

\[
\frac{dy}{dx} + \left( \frac{2x - 1}{1 + x^2} \right) y = 0,
\]

and the integrating factor is

\[
\exp \left( \int \frac{2x - 1}{1 + x^2} dx \right) = \exp(\ln(1 + x^2) - \tan^{-1} x) = (1 + x^2) e^{-\tan^{-1} x}.
\]

So we have

\[
\frac{d}{dx} \left( (1 + x^2) e^{-\tan^{-1} x} y \right) = 0
\]

which can be integrated to give

\[
(1 + x^2) e^{-\tan^{-1} x} y = C, \quad \text{or} \quad y = \frac{Ce^{\tan^{-1} x}}{1 + x^2}.
\]
2. In a prolific breed of rabbits, the birth and death rates are each proportional to the square of the population \( N \). Let \( k_1, k_2 \) be the constants of proportionality for births and deaths respectively. Assume \( k_1 > k_2 \) and write \( k = k_1 - k_2 \). Then \( \frac{dN}{dt} = kN^2 \), where \( k > 0 \), and \( t \) is the time in months. Solve this differential equation to show that

\[
N(t) = \frac{N_0}{1 - kN_0 t},
\]

where \( N_0 \) is the initial population.

Suppose that \( N_0 = 2 \) and that there are 4 rabbits after 3 months. What does this model predict will happen after another 3 months?

**Solution:** The equation \( \frac{dN}{dt} = kN^2 \) can be separated:

\[
\int \frac{1}{N^2} dN = k \int dt
\]

\[
-\frac{1}{N} = kt + C.
\]

If \( N = N_0 \) when \( t = 0 \) then substitution shows that \( C = -1/N_0 \). Hence the required particular solution is \( N = \frac{N_0}{1 - kN_0 t} \).

We are given that \( N_0 = 2 \), and the remaining unknown \( k \) may be found by using the further information that \( N = 4 \) when \( t = 3 \). Substitution then shows that \( 4 = \frac{2}{1 - 6k} \), or \( k = 1/12 \). Thus the population as a function of time in this case is \( N = \frac{2}{1 - t/6} \). After 6 months the model predicts an infinite rabbit population.

Only half that time earlier the population was just 4. At large \( N \) this simple model is no longer adequate.

3. Radiocarbon dating allows us to estimate the age of ancient objects. In living organisms, the ratio of radioactive carbon-14 to ordinary carbon-12 is constant. However when the organism dies, carbon-14 is no longer absorbed (from the atmosphere or via feeding for example), and so the amount present decreases through radioactive decay. By comparing the amount of carbon-14 present with the amount which would normally be present, we can determine the number of years since an organism died (or the age of objects such as clothing or paper, made from once-living material).

The half-life of carbon-14 is 5730 years. This means that the rate of decay is proportional to the amount \( y(t) \) of carbon-14 present,

\[
\frac{dy}{dt} = -ky \quad (k > 0),
\]

and that half of any given amount will disintegrate in 5730 years.
(a) Find the solution of this equation given that at time $t = 0$, the amount of carbon-14 present is $y_0$.

**Solution:** The equation is separable:

$$\int \frac{1}{y} \, dy = \int (-k) \, dt,$$

and thus we obtain $\ln y = -kt + c$, i.e. $y = Ae^{-kt}$ where $A = e^c$. Since $y(0) = y_0$, we have $y_0 = Ae^0 = A$. So $A = y_0$ and $y = y_0 e^{-kt}$.

(b) Use the fact that half of this amount $y_0$ will disintegrate over 5730 years to find a numerical value for $k$.

**Solution:** We are given that $y(5730) = \frac{1}{2}y_0$, and so $\frac{1}{2}y_0 = y_0 e^{-5730k}$, which then gives $k = (-1/5730) \ln(\frac{1}{2}) = 0.000121$ (to 6 decimal places).

(c) A piece of woollen clothing is found to have only 77% of the amount of carbon-14 normally found in wool. Estimate the age of this piece of clothing.

**Solution:** If only 77% of the original amount of carbon-14 remains, we must have $0.77y_0 = y_0 e^{-0.000121t}$ and so $t = (-1/0.000121) \ln(0.77) \approx 2160$ years.

4. A lucerne crop on an experimental farm is grown on an unirrigated paddock. The rate of growth of the crop depends on the mass of the lucerne plants and the water content of the soil. The water content of the soil declines exponentially with time when there is no rain. Once the soil water content falls below a critical level $W_c$ the crop stops growing and starts to die back.

The soil moisture content $M(t)$ is modelled as $M(t) = Ke^{-at}$.

(a) Sketch $M$ as a function of $t$.

**Solution:**

![Graph of $M$ vs $t$]

(b) Let $s$ be the mass of the crop in kilograms per hectare and let $t$ be the time in weeks. The following equation is proposed as a model for this crop’s growth:

$$\frac{ds}{dt} = m(Ke^{-at} - W_c)s \quad (m \text{ a constant}).$$

Explain why this equation might be a reasonable model for crop growth.

**Solution:** The rate of change $\frac{ds}{dt}$ is proportional to crop size $s$; $(Ke^{-at} - W_c)$ is the amount of soil moisture above the critical level $W_c$ and $\frac{ds}{dt}$ is proportional to this too; $m$ is a constant of proportionality. This fits all the known (important) facts about the crop.
(c) Find the time $t$ when the crop stops growing and starts to die back.

**Solution:** When the crop stops growing, $\frac{ds}{dt} = 0$. So $m(Ke^{-at} - W_c) = 0$, or $Ke^{-at} - W_c = 0$. Solving for $t$ yields $t = -\frac{1}{a} \ln(K/W_c)$.

(d) Find the general solution to the differential equation in part (b).

**Solution:** Separating the variables gives

$$\int \frac{ds}{s} = \int m(Ke^{-at} - W_c) dt$$

and integrating this gives

$$\ln |s| = m \left( -\frac{K}{a} e^{-at} - W_c t \right) + C ,$$

where $C$ is a constant.

Thus the general solution is $s = A \exp \left( m \left( -\frac{K}{a} e^{-at} - W_c t \right) \right)$ where $|A| = e^C$.

An agricultural scientist has data on this crop in this paddock from the previous seasons. He knows that $m = 0.035$, $W_c = 1.3$ and $a = 0.41$.

(e) A crop is planted and starts to grow. Then heavy rain falls and increases the soil moisture content to 15 times $W_c$, the critical level. Let $t = 0$ immediately after the rain. Use the expression for soil moisture $M(t)$ to find $K$.

**Solution:** When $t = 0$, $M = 15W_c$; so $15W_c = Ke^0$. Hence $K = 15W_c = 19.5$.

(f) How soon after the rain does the crop stop growing?

**Solution:** The crop stops growing when $t = -\frac{1}{a} \ln(K/W_c)$ (from part (c)); that is, when $t = 6.6$. The crop stops growing 6.6 weeks after the rain.

(g) When the rain falls the crop has grown to a mass of 400 kg per hectare. What is the maximum mass per hectare attained by the crop if there is no further rain? If the crop is cut 12 weeks after the rain what will be the yield per hectare?

**Solution:** If $s = 400$ when $t = 0$ then

$$400 = A \exp \left( -\frac{mK}{a} \right) , \quad \text{or} \quad A = 400 \exp \left( \frac{mK}{a} \right) \approx 2113 .$$

Therefore at time $t$ the mass is per hectare is

$$s = 2113 \exp \left( 0.035 \left( -47.5e^{-0.41t} - 1.3t \right) \right) .$$

The mass per hectare is at a maximum when $t = 6.6$. Substituting this into the equation for $s$ we get $s = 1398$. So the crop attains a maximum of 1400 kg per hectare (correct to 3 significant figures).

When $t = 12$, $s = 1210$. So 12 weeks after the rain, when the crop is cut, the yield will be 1210 kg per hectare.
5. A holding pond at a chemical treatment plant contains 2000 litres of water which has been contaminated with a particular impurity. The concentration of this impurity in the pond is initially 5 grams per litre. Water is drained from the pond for treatment at a rate of 10 litres per minute and water from another pond is added at the same rate. This input water also contains the impurity at a concentration of 0.02 grams per litre. A safe level of this impurity is considered to be 1 gram per litre. Formulate a differential equation for the rate of change of the mass of impurity in the holding pond. Assuming that the water in the holding pond is well-stirred, how long does it take for the impurity to reach safe levels?

Solution: Let $I(t)$ be the mass in grams of impurity in the pond. Because inflow and outflow rates are equal, the volume of water in the tank remains constant at 2000 litres. Let $t$ be the time in minutes. The loss of impurity through draining is $10I/2000$ grams/minute = $0.005I$. The input of impurity through input water is $10 \times 0.02 = 0.2$ grams per minute. So

$$\frac{dI}{dt} = -\frac{10I}{2000} + 0.2 = -0.005I + 0.2.$$ 

Separating and integrating $\int dt = \int \frac{dI}{-0.005I + 0.2}$ gives

$$t + C = \frac{-1}{0.005} \ln|0.2 - 0.005I|, \quad \text{or} \quad I = 200(0.2 - Ae^{-0.005t}).$$

When $t = 0$, $I = 5 \times 2000 = 10,000$ grams so $10000 = 200(0.2 - A)$ and $A = -49.8$. Hence $I = 200(0.2 + 49.8e^{-0.005t})$.

When the impurity reaches safe levels in the pond, $I = 2000$ grams, and so $2000 = 200(0.2 + 49.8e^{-0.005t})$.

Thus $49.8e^{-0.005t} = 9.8$, which then gives $-0.005t = \ln\left(\frac{9.8}{49.8}\right)$, and so

$$t = 325 \text{ minutes} \quad \text{or} \quad t = 5 \text{ hours 25 minutes}.$$