Assumed Knowledge: Finding the roots of quadratic equations. Euler’s formula
\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

Objectives:

(11a) To be able to write down the auxiliary (or characteristic) equation associated with
a second-order differential equation with constant coefficients.

(11b) To be able to construct the solutions to such differential equations in terms of
exponential and trigonometric functions.

Exercises:

1. Find the general solution of \( \frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 4y = 0. \)

   **Solution:** The auxiliary equation is \( \lambda^2 + 2\lambda - 4 = 0 \) so that \( \lambda = -1 \pm \sqrt{5}. \)

   Hence \( y = Ae^{(-1+\sqrt{5})t} + Be^{(-1-\sqrt{5})t} = e^{-t}(Ae^{\sqrt{5}t} + Be^{-\sqrt{5}t}). \)

2. Find the particular solution of \( \frac{d^2y}{dt^2} - 9y = 0 \) which satisfies the initial conditions
   \( y = 3 \) and \( \frac{dy}{dt} = 3 \) when \( t = 0. \)

   **Solution:** The auxiliary equation is \( \lambda^2 - 9 = 0 \) so that \( \lambda = 3 \) or \( \lambda = -3. \)

   The general solution is therefore \( y = Ae^{3t} + Be^{-3t}. \)

   Now, \( \frac{dy}{dt} = 3Ae^{3t} - 3Be^{-3t}, \) and so when \( t = 0, \) \( y = A + B = 3 \)

   and \( \frac{dy}{dt} = 3A - 3B = 3. \) Solving these equations gives \( A = 2 \) and \( B = 1, \) so the particular

   solution is \( y = 2e^{3t} + e^{-3t}. \)

3. Find the general solutions of these second-order homogeneous equations. In each
   case give your answer in terms of real functions.

   (a) \( \frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 0. \)

      **Solution:** Auxiliary equation is \( 2\lambda^2 - 7\lambda + 3 = 0 \) or \( (2\lambda - 1)(\lambda - 3) = 0. \)

      So \( \lambda = \frac{1}{2}, 3. \)

      So the general solution is \( y = Ae^{x/2} + Be^{3x}. \)

   (b) \( \frac{d^2y}{dx^2} + 3y = 0. \)

      **Solution:** Auxiliary equation is \( \lambda^2 + 3 = 0. \) So \( \lambda = \pm i\sqrt{3}. \)

      So the general solution is \( y = A \cos \sqrt{3}x + B \sin \sqrt{3}x. \)
Solution: Auxiliary equation is $\lambda^2 - 2\lambda + 2 = 0$. So $\lambda = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$.
Therefore the general solution is $y = e^x (A \cos x + B \sin x)$.

4. Find the general solution of each of these second-order linear homogeneous equations. Hence find the particular solution for the given conditions. In each case give your answer in terms of real functions.

(a) $\frac{d^2 y}{dx^2} + dy = 0$, $y(0) = y'(0) = 1$.
Solution: Auxiliary equation is $\lambda^2 + \lambda - 20 = 0$ or $(\lambda + 5)(\lambda - 4) = 0$. So $\lambda = 4, -5$.
General solution is $y = Ae^{-5x} + Be^{4x}$.
When $x = 0$, $y = 1$ so $A + B = 1$.
Now $y' = \frac{dy}{dx} = -5Ae^{-5x} + 4Be^{4x}$. When $x = 0$, $y' = 1$ so $-5A + 4B = 1$.
Hence $B = \frac{2}{3}$, $A = \frac{1}{3}$.
Particular solution is $y = \frac{1}{3}e^{-5x} + \frac{2}{3}e^{4x}$.

(b) $\frac{d^2 y}{dx^2} + 9y = 0$, $y(0) = 1$, $y(\pi/6) = 3$.
Solution: Auxiliary equation is $\lambda^2 + 9 = 0$, so $\lambda = \pm 3i$.
General solution is $y = E \cos 3x + F \sin 3x$.
When $x = 0$, $y = 1$ so $E = 1$.
When $x = \frac{\pi}{6}$, $y = 3$ so $F = 3$.
Particular solution is $y = \cos 3x + 3 \sin 3x$.

(c) $\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 7y = 0$, $y(0) = 0$, $\frac{dy}{dt} = 3$ when $t = 0$.
Solution: Auxiliary equation is $\lambda^2 + 4\lambda + 7 = 0$. So $\lambda = \frac{-4 \pm \sqrt{16 - 28}}{2} = -2 \pm \sqrt{3}i$.
General solution is $y = e^{-2t}(E \cos \sqrt{3}t + F \sin \sqrt{3}t)$.
When $t = 0$, $y = 0$ so $E = 0$.
When $t = 0$, $\frac{dy}{dt} = 3$ so calculate $\frac{dy}{dt} = -2e^{-2t}F \sin \sqrt{3}t + \sqrt{3}e^{-2t}F \cos \sqrt{3}t$, since $E = 0$.
At $t = 0$, we have $3 = \sqrt{3}F$ so $F = \sqrt{3}$.
The particular solution is thus $y = \sqrt{3}e^{-2t} \sin \sqrt{3}t$.

(d) $\frac{d^2 x}{dt^2} - 4 \frac{dx}{dt} + 4x = 0$, $x(0) = 1$, $x(1) = 3e^2$.
Solution: Auxiliary equation is $\lambda^2 - 4\lambda + 4 = 0$ or $(\lambda - 2)^2 = 0$
Since $\lambda = 2$ is a repeated root, the general solution is $x = Ae^{2t} + Bte^{2t}$.
When $t = 0$, $x = 1$ so $A = 1$.
When $t = 1$, $x = 3e^2$ so $3e^2 = e^2 + Be^2$ or $B = 2$,
The particular solution is $x = e^{2t} + 2te^{2t}$. 

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