The University of Sydney

School of Mathematics and Statistics

Summer School 2012
SS1003 Assignment 1

The assignment should be written in ink on **one side of the paper** only and **stapled** into a manila folder on the cover of which is written your name and SID.

It is due on Tuesday 17th January and should be handed in at the start of your tutorial on that day. It is intended to provide feedback: it will be qualitatively marked and returned the following Monday. It will not contribute to the assessment for the subject.

Q1 (a) Find the Lower (L_6) and Upper (U_6) Riemann Sums for the definite integral

$$\int_2^5 x^2 dx$$

(b) What is the smallest value of N for which $U_N - L_N < \frac{1}{10}$?

Q2 (a) Sketch the area bounded by the curves $y = \cos x$, $y = \cos 2x$ and the line $x = \pi$ between $x = 0$ and $x = \pi$

(b) Where do the 2 curves intersect? (Hint: $\cos 2x = 2\cos^2 x - 1$)

(c) Find the bounded area.

Q3 (a) Sketch the area in the xy-plane bounded by the curves $y = x^2$, $y = x^3$ and the line $x = 2$ between $x = 1$ and $x = 2$

(b) Find the volume of the solid of revolution formed when this region is rotated about the axis $x = -1$

(c) Find the volume of the solid of revolution formed when this region is rotated about the x axis

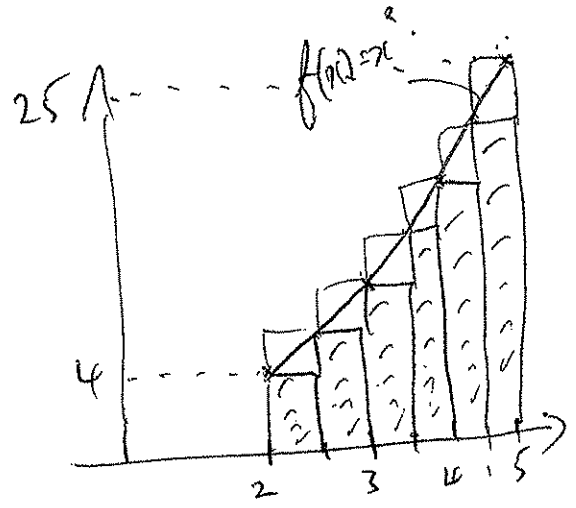
Q4 Evaluate the following integrals by making a substitution

(a) $\int_0^1 \frac{x}{\sqrt{2-x^2}}\,dx$ (b) $\int_{\frac{\pi}{2}}^{\pi} \sin^3 x\,dx$

Q5 Use an appropriate integral and Riemann sums to show that

$$7500 < \sum_{n=1}^{1000} n^{\frac{1}{3}} < 7510\,.$$

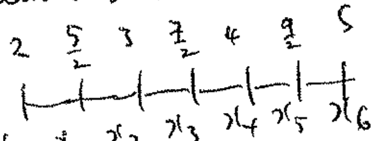
Q1 (a)



$$\int_2^5 x^2\,dx$$

Partition into 6 subintervals of width $\Delta x = \frac{5-2}{6} = \frac{1}{2}$

The function x^2 is strictly increasing so the lower bound L_6 is obtained by choosing the left hand end of each sub-interval.



$$L_6 = \left(f(x_0)+f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)\right)\Delta x = \sum_{i=0}^{5} f(x_i)\Delta x$$

$$= \left(2^2+\left(\tfrac{5}{2}\right)^2+3^2+\left(\tfrac{7}{2}\right)^2+4^2+\left(\tfrac{9}{2}\right)^2\right)\frac{1}{2}$$

$$\boxed{L_6 = \frac{4^2+5^2+6^2+7^2+8^2+9^2}{8}} \qquad \left(=\frac{271}{8}=33.875\right)$$

the upper sum is chosen by choosing Right Hand end of each subinterval

$$U_6 = \sum_{i=1}^{6} f(x_i)\Delta x = \boxed{\frac{5^2+6^2+7^2+8^2+9^2+10^2}{8}} = U_6$$

$$= \left(\frac{355}{8} = 44.375\right)$$

Q1(b) for a partition with N equal subintervals

$$\Delta x = \frac{5-2}{N} = \frac{3}{N}$$

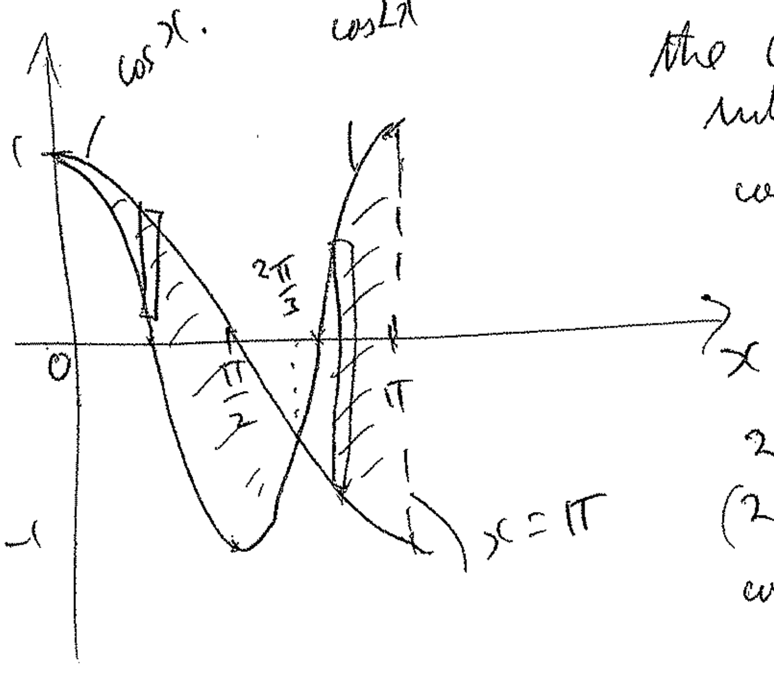
the difference $U_N - L_N = \left(f(5)-f(2)\right)\Delta x$.

$$U_N - L_N = (25-4)\times\frac{3}{N} = \frac{63}{N}$$

if $U_N - L_N < \frac{1}{10}$ $\qquad \frac{63}{N} < \frac{1}{10}$.

$$\boxed{N > 630}$$

Q2



the curves intersect where

$\cos x = \cos 2x$

$= 2\cos^2 x - 1$

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = -\frac{1}{2}$, $\cos x = 1$

$x = \frac{2\pi}{3}$; $x = 0$,

Q2 continued

between $x = 0$ and $x = \frac{2\pi}{3}$ $\cos x > \cos 2x$

so contribution to the area $= \int_0^{\frac{2\pi}{3}} (\cos x - \cos 2x)\, dx$

between $x = \frac{2\pi}{3}$ and $x = \pi$ $\cos 2x > \cos x$

so contribution to the area $= \int_{\frac{2\pi}{3}}^{\pi} (\cos 2x - \cos x)\, dx$

total area bounded

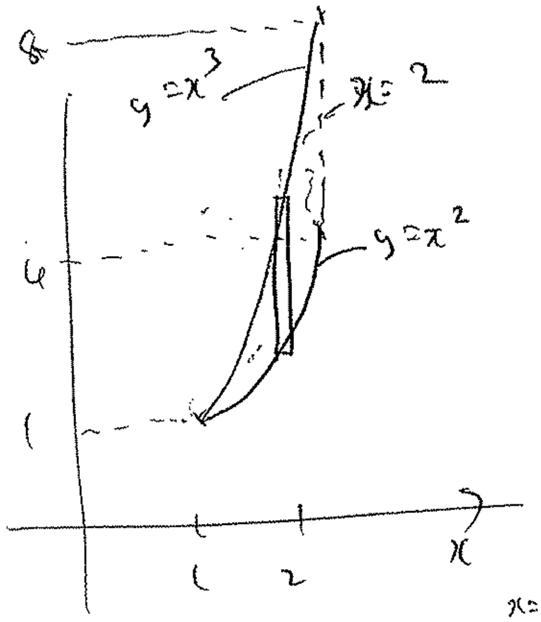
$$= \left[\sin x - \frac{1}{2}\sin 2x \right]_0^{\frac{2\pi}{3}} + \left[\frac{1}{2}\sin 2x - \sin x \right]_{\frac{2\pi}{3}}^{\pi}$$

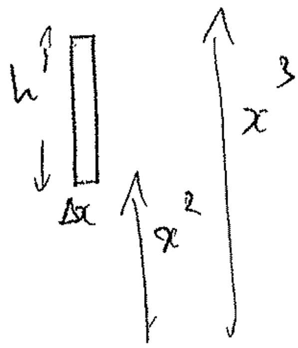
$$= \left(\sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3} \right) - 0 + 0 - \left(\frac{1}{2}\sin\frac{4\pi}{3} - \sin\frac{2\pi}{3} \right)$$

$$= 2\left(\sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3} \right) \quad \boxed{= \frac{3\sqrt{3}}{2}} \quad (2.598)$$

area bounded is $\frac{3\sqrt{3}}{2}$ square units.

(Q3) (4)





(a) $\Delta A \simeq h\Delta x. \therefore A = \int_{x=1}^{x=2} h(x)\,dx.$

$x^3 \geq x^2$ for $x \geq 1$

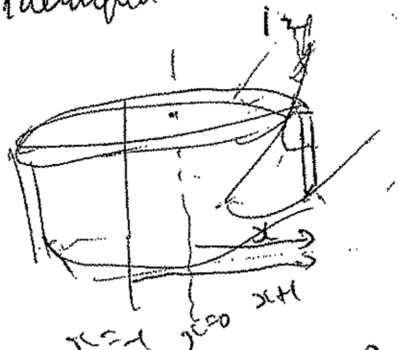
$h = x^3 - x^2$

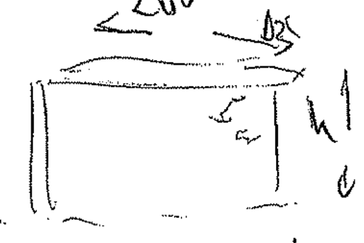
$$A = \int_{x=1}^{2} (x^3 - x^2)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2$$

$$= \frac{2^4}{4} - \frac{2^3}{3} - \left(\frac{1^4}{4} - \frac{1^3}{3}\right) \quad \boxed{= \frac{17}{12}}$$

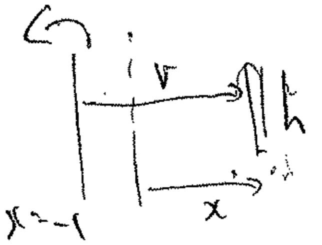
area identified is 17 units squared.

(b)





$\Delta V \simeq 2\pi r h \Delta x.$



$$V = \int_1^2 2\pi r(x) h(x)\,dx$$

$r(x) = x+1$; $h(x) = x^3 - x^2$

$$= \int_1^2 2\pi (x+1)(x^3 - x^2)\,dx.$$

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3(b) cont'd

$$2\pi\int_1^2 (x^4 - x^3 + x^3 - x^2)\,dx$$

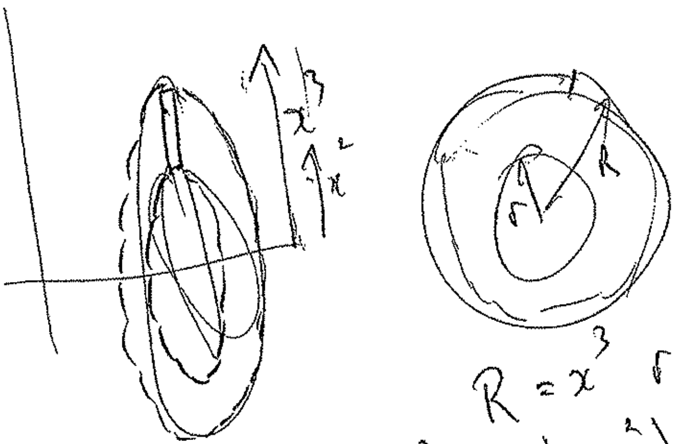
$$= 2\pi \int_1^2 (x^4 - x^2)\,dx$$

$$= 2\pi \left[\frac{x^5}{5} - \frac{x^3}{3} \right]_1^2$$

$$= 2\pi \left(\left(\frac{32}{5} - \frac{8}{3} \right) - \left(\frac{1}{5} - \frac{1}{3} \right) \right)$$

$$= 2\pi \left(\frac{31}{5} - \frac{7}{3} \right) \quad \boxed{= \frac{116\pi}{15}} \quad (24.295)$$

3c



$\Delta V = A\,\Delta x$

$A = (\pi R^2 - \pi r^2)$

$R = x^3 \quad r = x^2$

$$V = \int_1^2 \pi \left((x^3)^2 - (x^2)^2 \right) dx$$

$$= \pi \int_1^2 (x^6 - x^4)\,dx$$

$$= \pi \left[\frac{x^7}{7} - \frac{x^5}{5} \right]_1^2$$

$$\boxed{= \frac{418\pi}{35}} \quad (37.52)$$

Q4 (a) $\int_0^1 \frac{x}{\sqrt{2-x^2}}\,dx$

consider $\int \frac{x}{\sqrt{2-x^2}}\,dx$. $\rightarrow$ $-\frac{1}{2}\int \frac{-2x}{\sqrt{2-x^2}}\,dx$.

let $u = 2-x^2$

then $u'(x) = -2x$.

$$-\frac{1}{2}\int \frac{-2x}{\sqrt{2-x^2}}\,dx$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}} u'\,dx$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= -\frac{1}{2} \times 2u^{\frac{1}{2}}$$

$$= -u^{\frac{1}{2}}$$

$$= -\sqrt{2-x^2}$$

so $\int_0^1 \frac{x}{\sqrt{2-x^2}}\,dx = \left[-\sqrt{2-x^2}\right]_{x=0}^{x=1}$

$$= \left(-\sqrt{2-1^2}\right) - \left(-\sqrt{2-0^2}\right)$$

$$\boxed{= \sqrt{2} - \sqrt{1}} \quad \boxed{= \sqrt{2} - 1}$$

$\left(\text{or } = \int_{u=2}^{u=1} -u^{\frac{1}{2}}\,du \right)$

(6b) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \sin^3 x\,dx$: first consider (7)

$$\int \sin^3 x\,dx$$

$$= \int \sin^2 x \sin x\,dx.$$

$$= -\int (1-\cos^2 x)(-\sin x)\,dx$$

let $u = \cos x$. then $u' = -\sin x$

$$-\int (1-u^2)\underbrace{u'\,dx}_{du}$$

$$= \int (u^2-1)\,du$$

$$= \frac{u^3}{3} - u$$

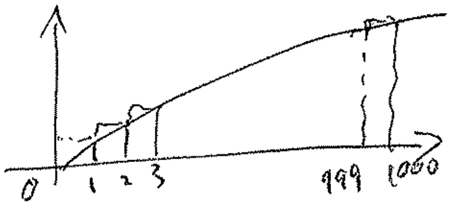
$$= \frac{\cos^3 x}{3} - \cos x$$

so $$\int_{x=\frac{\pi}{2}}^{\pi} \sin^3 x\,dx = \left[\frac{\cos^3 x}{3} - \cos x\right]_{x=\frac{\pi}{2}}^{x=\pi}.$$

$$= \frac{\cos^3 \pi}{3} - \cos\pi - \frac{\cos^3 \frac{\pi}{2}}{3} + \cos\frac{\pi}{2}$$

$$= \frac{(-1)^3}{3} - (-1) - 0, = \boxed{\frac{2}{3}}$$

Q5. Consider $\int_0^{1000} x^{\frac{1}{3}} dx$



Partition the interval $[0, 1000]$ into 1000 equal subintervals.
Because $x^{\frac{1}{3}}$ is strictly increasing on the interval
the smallest function value is given by the left hand function value in each subinterval and the largest by the right hand value.

hence $L_{1000} = \sqrt[3]{0} + \sqrt[3]{1} \cdots + \sqrt[3]{999} = \sum_{n=0}^{999} \sqrt[3]{n}$

and $U_{1000} = \sqrt[3]{1} + \sqrt[3]{2} + \cdots + \sqrt[3]{1000} = \sum_{n=1}^{1000} \sqrt[3]{n}$

also $\int_0^{1000} x^{\frac{1}{3}} dx = \frac{3}{4} x^{\frac{4}{3}} \Big]_{x=0}^{x=1000} = \frac{3}{4}(10{,}000 - 0)$
$= 7500$.

Hence $L_{1000} < \boxed{7500 < U_{1000}}$ ①

consider $L_{1000} < 7500$ and add $\sqrt[3]{1000}$ to both sides.

$\therefore \sum_{n=0}^{999} \sqrt[3]{n} + \sqrt[3]{1000} < 7500 + \sqrt[3]{1000}$

ie $\boxed{\sum_{n=1}^{1000} \sqrt[3]{n} < 7510}$ ②

combining the 2 inequalities.

$7500 < \sum_{n=1}^{1000} \sqrt[3]{n} < 7510$ as required.