The University of Sydney

School of Mathematics and Statistics

Summer School 2012
SS1003 Assignment 2

The assignment should be written in ink on one side of the paper only and stapled into a manila folder on the cover of which is written your name and SID.

It is due on Monday 30th January and should be handed in at the start of the tutorial on that day. It is intended to provide feedback: it will be marked and returned the next day. It will not contribute to the assessment for the subject.

Q1 Define $G(x)$ by $G(x) = \int_1^x (\ln t)^2 \, dt$

(a) What is the derivative $\frac{dG}{dx}$?

(b) Define $H(x)$ by $H(x) = \int_1^{x^2} (\ln t)^2 \, dt$: Express $H(x)$ in terms of the function G

(c) What is the derivative $\frac{dH}{dx}$?

Q2 Find $\frac{dy}{dx}$ for the following functions

(a) $y = e^{x^2 + 2\ln x}$ (b) $y = \ln \frac{\sin x}{\cos x}$ (c) $y = x^{x^2}$ (d) $y = (x^2)^x$

(e) $y = \log_3 x^3$ (f) $y = 3^{\ln x}$ (g) $y = \frac{x^3 \cos^2 3x}{e^{\sin x}(5x^2 + x - 1)^3}$

Q3 Find the general solution of the following differential equations and in each case then find the particular solution corresponding to the given condition

(a) $\frac{dy}{dx} = x \sin x^2$: $y(0) = \frac{1}{2}$ (b) $\frac{dx}{dt} = \ln t$: $x(1)=0$

Q4 Find the following indefinite integrals using integration by parts

(a) $\int t \ln t \, dt$ (b) $\int t^2 \cos 3t \, dt$ (c) $\int e^x \cos 2x \, dx$

SS1003 ASSIGNMENT 2 SOLUTIONS (V2) (1)

Q1

(a) $\frac{dG}{dx} = (\ln x)^2$ (using fundamental theorem)

(b) $H(x) = G(x^2)$

(c) $\frac{dH}{dx} = \frac{dG(x^2)}{dx^2}\frac{dx^2}{dx}$ using chain rule $= 2x(\ln x^2)^2$

Q2 (a) $y = e^{x^2+2\ln x}$

$$\frac{dy}{dx} = e^{x^2+2\ln x} \times \frac{d}{dx}\left(x^2+2\ln x\right)$$

$$= 2\left(x+\frac{1}{x}\right)e^{x^2+2\ln x}$$

$$= 2x^2\left(x+\frac{1}{x}\right)e^{x^2}$$

$$= 2x(x^2+1)e^{x^2}$$

b $y = \ln \frac{\sin x}{\cos x}$

$= \ln \sin x - \ln \cos x$

$\frac{dy}{dx} = \frac{1}{\sin x} \frac{d \sin x}{dx} - \frac{1}{\cos x} \frac{d \cos x}{d x}$

$= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$

$= \cot x + \tan x$

c $y = x^{x^2}$

$\Rightarrow \ln y = x^2 \ln x$

$\frac{d \ln y}{dx} = \frac{1}{y} y' = 2x \ln x + \frac{x^2}{x} = 2x \ln x + x$

$y' = x^{x^2}(x + 2x \ln x)$

$= x^{x^2+1}(1 + 2 \ln x)$

d $y = (x^2)^x = x^{2x}$

$\ln y = 2x \ln x$

$\frac{1}{y} y' = \frac{d}{dx}(2x \ln x) = 2 \ln x + 2$

$y' = 2x^{2x}(\ln x + 1)$

(3)

(e) $y = \log_3 x^3 = \frac{\ln x^3}{\ln 3}$

$$y = \frac{3\ln x}{\ln 3}$$

$$\frac{dy}{dx} = \frac{3}{x\ln 3}$$

(g) $y = \frac{x^3 \cos^2 3x}{e^{\sin x}(5x^2+x-1)^3}$

take ln of each side

$$\ln y = 3\ln x + 2\ln\cos 3x - \sin x - 3\ln(5x^2+x-1)$$

differentiate wrt x

$$\frac{1}{y}y' = \frac{3}{x} + \frac{2(-3\sin 3x)}{\cos 3x} - \cos x - \frac{3(10x+1)}{5x^2+x-1}$$

$$y' = \frac{x^3\cos^2 3x}{e^{\sin x}(5x^2+x-1)^3}\left(\frac{3}{x} - 6\tan 3x - \cos x - \frac{3(10x+1)}{5x^2+x-1}\right)$$

(f) $\frac{d}{dx}\left(3^{\ln x}\right) = \frac{d}{d(\ln x)} 3^{(\ln x)} \times \frac{d(\ln x)}{dx}$

$$= (\ln 3)3^{\ln x} \times \frac{1}{x}$$

$$= \frac{\ln 3}{x} 3^{\ln x}$$

Q3 (a) $\frac{dy}{dx} = x\sin x^2$

$\Rightarrow y = \int x \sin x^2 \, dx + C$

let $x^2 = u. \rightarrow u' = 2x.$

$y = \frac{1}{2}\int 2x \sin x^2 \, dx + C$

$= \frac{1}{2}\int \sin u \, u' \, dx + C$

$= \frac{1}{2}\int \sin u \, du + C$

$= -\frac{1}{2}\cos u + C$

$= -\frac{1}{2}\cos x^2 + C$ general solution.

$y(x) =$

when $x = 0$ $y = \frac{1}{2}$. $-\frac{1}{2}\cos 0 + C \Rightarrow c = 1.$

$y(0) = c - \frac{1}{2}$

$\frac{1}{2} =$ $1 - \frac{1}{2}\cos x^2$ particular solution

$y(x) =$

(b) $\frac{dx}{dt} = \ln t \Rightarrow x = \int \ln t \, dt + C$

$= \int 1 \ln t \, dt + C$

use integration by parts let $u = \ln t$ $v' = 1 \Rightarrow u' = \frac{1}{t}$ $v = t$.

$= t\ln t - \int t \cdot \frac{1}{t} \, dt + C$

$x(t) = t\ln t - t + C$ general solution

$x(1) = 0 = 1\ln 1 - 1 + C$

$= 0 - 1 + C \Rightarrow c = 1$

$x(t) = t\ln t - t + 1$ particular solution

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Q4 (a) $\int t \ln t \, dt$

let $u = \ln t$, $v' = t$.

then $u' = \frac{1}{t}$, $v = \frac{t^2}{2}$ $\quad uv - \int u'v \, dt$

$$\int t \ln t \, dt = \frac{t^2}{2} \ln t - \int \frac{1}{t} \frac{t^2}{2} dt$$

$$= \frac{t^2}{2} \ln t - \int \frac{t}{2} dt$$

$$\boxed{= \frac{t^2}{2} \ln t - \frac{t^2}{4}}$$

(b) $\int t^2 \cos 3t \, dt$

let $u = t^2$, $v' = \cos 3t \Rightarrow u' = 2t$, $v = \frac{1}{3} \sin 3t$,

$$uv - \int u'v \, dt = \frac{1}{3} t^2 \sin 3t - \int \frac{2}{3} t \sin 3t \, dt.$$

integrate by parts again.

consider $\int t \sin 3t \, dt$.

let $r = t$; $s' = \sin 3t \Rightarrow r' = 1$, $s = -\frac{1}{3} \cos 3t$

$$\int t \sin 3t \, dt = -\frac{t}{3} \cos 3t + \int \frac{1}{3} \cos 3t \, dt$$

$$= -\frac{t}{3} \cos 3t + \frac{\sin 3t}{9}.$$

so $\int t^2 \cos 3t \, dt = \boxed{\frac{1}{3} t^2 \sin 3t + \frac{2}{3} \left[\frac{t}{3} \cos 3t - \frac{\sin 3t}{9} \right]}$

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4c) $\int e^x \cos 2x \, dx$ $\qquad uv - \int u'v \, dx$

let $e^x = v'$ let $u = \cos 2x$.

then $v = e^x$ $u' = -2\sin 2x$.

$= e^x \cos 2x - \int (-2\sin 2x) e^x \, dx$.

$= e^x \cos 2x + 2\int \sin 2x \, e^x \, dx$

let $r' = e^x$ $\quad s = \sin 2x$.

$r = e^x$ $\quad s' = 2\cos 2x$.

$\int \sin 2x \, e^x \, dx = rs - \int r's \, dx$

$= e^x \sin 2x - \int e^x 2\cos 2x \, dx$

$=$

so $\int e^x \cos 2x \, dx = e^x \cos 2x + 2\left(e^x \sin 2x - \int e^x 2\cos 2x \, dx \right)$

let $\int e^x \cos 2x \, dx = I$.

then $I = e^x(\cos 2x + 2\sin 2x) - 4I$.

$5I = e^x(\cos 2x + 2\sin 2x)$

$\Rightarrow I = \int e^x \cos 2x \, dx = \boxed{\frac{e^x}{5}(\cos 2x + 2\sin 2x)}$