The University of Sydney

School of Mathematics and Statistics

Summer School 2011
SS1003 Assignment 3

The assignment should be written in ink on one side of the paper only and stapled into a manila folder on the cover of which is written your name and SID.

It is due on Tuesday 8th February and should be handed in at your tutorial on that day. It is intended to provide feedback: it will be marked and returned the following Monday. It will not contribute to the assessment for the subject.

Q1 Find the general solution of

(a) $\ln y \frac{dy}{dx} = xy$ (b) $\frac{dy}{dx} = \frac{\tan y}{\sqrt{x^2+9}}$ (c) $\frac{dy}{dx} = \frac{e^{2x}}{\sqrt{1+e^x}}$ Hint: in (c) make a substitution

2 Evaluate the integrals

(a) $\int \frac{3x-2}{(x+1)(x-2)}dx$ (b) $\int \frac{x^3}{(x+1)(x-2)}dx$ (c) $\int \frac{x^2+1}{(x+1)(x-1)^2}dx$

Q3 The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What was the value of N after 1.5 hours?

Q4 The number x in a population satisfies the logistic equation

$$\frac{dx}{dt} = 10x(20-x) \text{ where } t \text{ is the time in days}$$

(a) What is the equilibrium population?

(b) If the population is initially 40, how long does it take for it to reach 30?

Q5 Determine the particular solution curve y(x) of the differential equation

$$\frac{dy}{dx} = -\frac{3y}{x} - \frac{2}{x^4}$$

which passes through the point (e^2, e^{-6}).

SUGGESTED SOLUTIONS TO ASSIGNMENT 3

Q1(a) $\ln y \frac{dy}{dx} = xy$ separates $\frac{1}{y} \ln y \frac{dy}{dx} = x$

$$\int \frac{1}{y} \ln y \, dy = \int x \, dx + C$$

make sub $u = \ln y$.
then $u' = \frac{1}{y}$

$$\int u \, du = \frac{u^2}{2} \Rightarrow \boxed{\frac{(\ln y)^2}{2} = \frac{x^2}{2} + C}$$

(which simplifies to $y = e^{\sqrt{x^2+2C}}$)

Q1b separates $\int \frac{\cos y}{\sin y} dy = \int \frac{1}{\sqrt{x^2+9}} dx + C$

(make sub $u = \sin y$
$\int \frac{1}{u} du = \ln|u|$)

$$\boxed{\ln|\sin y| = \sinh^{-1}\frac{x}{3} + C}$$

from table

Q1c $y = \int \frac{e^{2x} \, dx}{\sqrt{1+e^x}}$ let $u = 1+e^x \Rightarrow e^x = u-1$.
$x = \ln(u-1)$
$\frac{dx}{du} = \frac{1}{u-1}$

$$y = \int \frac{(u-1)^2}{\sqrt{u}} \frac{1}{(u-1)} du$$

$$= \int \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right) du = \frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C.$$

$$\boxed{y = \frac{2}{3}(1+e^x)^{\frac{3}{2}} - 2(1+e^x)^{\frac{1}{2}} + C}$$

Q2 (a) use partial fractions to express

$$\frac{3x-2}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$$

$$3x-2 = A(x-2) + B(x+1)$$

put $x = 2$: $4 = 3B \Rightarrow B = \frac{4}{3}$

put $x = -1$ $-5 = -3A \Rightarrow A = \frac{5}{3}$

$$\frac{1}{3}\int\left(\frac{5}{x+1} + \frac{4}{x-2}\right)dx$$

$$= \boxed{\frac{1}{3}\left(5\ln|x+1| + 4\ln|x-2|\right) + C}$$

(2b) first divide by denominator

$$\begin{array}{r|l} & x+1 \\ x^2-x-2 & x^3 \\ & x^3-x^2-2x \\ \hline & x^2+2x \\ & x^2-x-2 \\ \hline & 3x+2 \end{array}$$

so $\frac{x^3}{(x+1)(x-2)}$

$$= x+1 + \frac{3x+2}{(x+1)(x-2)}$$

then express $\frac{3x+2}{(x+1)(x-2)}$ using partial fractions

$$= \frac{C}{(x+1)} + \frac{D}{(x-2)} \quad : \quad 3x+2 = C(x-2) + D(x+1)$$

$x: 2 \Rightarrow$ $8 = 3D \Rightarrow D = \frac{8}{3}$

$x = -1 \Rightarrow$ $-1 = -3C \Rightarrow C = \frac{1}{3}$

$$\int\left(x+1 + \frac{1}{3}\left(\frac{1}{x+1} + \frac{8}{x-2}\right)\right)dx = \boxed{\frac{x^2}{2} + x + \frac{1}{3}\left(\ln|x+1| + 8\ln|x-2|\right)}$$

Q2c write $\dfrac{x^2+1}{(x+1)(x-1)^2}$

in the form $\dfrac{A}{x+1} + \dfrac{B}{(x-1)} + \dfrac{C}{(x-1)^2}$

then $x^2+1 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$

put $x=1$ $\quad 2 = 2C \Rightarrow C = 1$

put $x=-1$ $\quad 2 = 4A \Rightarrow A = \frac{1}{2}$

equate coeff of x^2: $\quad 1 = A + B \Rightarrow B = \frac{1}{2}$

$$\frac{1}{2}\int\left(\frac{1}{(x+1)} + \frac{1}{(x-1)} + \frac{2}{(x-1)^2}\right)dx$$

$$= \boxed{\frac{1}{2}\left(\ln|x+1| + \ln|x-1| - \frac{2}{(x-1)}\right) + C}$$

Q3 let $N(t)$ be the population at time t hours

$\frac{dN}{dt} = kN$ which has solution

$$N(t) = N_0 e^{kt}$$

where $N(0) = N_0$

$$N(t) = 100 e^{kt}$$

$$N(1) = 100 e^{k \times 1} = 332$$

$$\Rightarrow e^{k} = \frac{332}{100}$$

we require $N\left(\frac{3}{2}\right) = 100 e^{\frac{3k}{2}}$

$$= 100 \left(e^{k}\right)^{\frac{3}{2}}$$

$$= 100 \left(\frac{332}{100}\right)^{\frac{3}{2}} \approx 605$$

after 1.5 hours the value of N was 605.

Q4 $\frac{dx}{dt} = 10x(20-x)$

The equilibrium population occurs when $\frac{dx}{dt} = 0$ ie at $x = 20$.

the equation is separable

$$\int \frac{1}{x(x-20)}\,dx = -\int 10\,dt$$

P.F. $\frac{1}{x(x-20)} = \frac{A}{x} + \frac{B}{(x-20)} \Rightarrow 1 = (x-20)A + Bx$

$x=0 \Rightarrow A = -\frac{1}{20}$ $x=20 \Rightarrow B = \frac{1}{20}$

$$\frac{1}{20}\int\left(\frac{1}{x-20} - \frac{1}{x}\right)dx = -\int 10\,dt + c$$

$$\ln\left|\frac{x-20}{x}\right| = -200t + c$$

$$\frac{x-20}{x} = Be^{-200t}$$

when $t=0$ $x=40 \Rightarrow \frac{40-20}{40} = \frac{1}{2} = Be^{0} = B$

when $x=30$ $\frac{30-20}{30} = \frac{1}{3} = \frac{1}{2}e^{-200t}$

$\Rightarrow t = \frac{1}{200}\ln\frac{3}{2}$ days ≈ 2.92 minutes

The population reaches 30 after approximately 2.92 minutes.

(5) $$\frac{dy}{dx} = -\frac{3y}{x} - \frac{2}{x^4}$$

FOL for $y = y(x)$

$$y' + \frac{3}{x}y = -\frac{2}{x^4}$$

$p(x) = \frac{3}{x} \rightarrow \int p\,dx = \int \frac{3}{x}\,dx = 3\ln x = \ln x^3$.

I.F. $r(x) = e^{\int p\,dx} = e^{\ln x^3} = x^3$

rewrite as $$\frac{d}{dx}(x^3 y) = -\frac{2}{x^4} \times x^3 = \frac{-2}{x}$$

solve $$x^3 y = -\int \frac{2}{x}\,dx + C$$

$$= -2\ln|x| + C$$

when $x = e^2$ $y = e^{-6}$ so.

$$(e^2)^3 e^{-6} = -2\ln|e^2| + C$$

$$1 = -4 + C \Rightarrow C = 5$$

$$x^3 y = 5 - 2\ln|x|$$

$$\boxed{y = \frac{5 - 2\ln|x|}{x^3}}$$