This tutorial covers material from the lectures in Day 2.

1. A first year mathematics student used the Fundamental Theorem of Calculus to do the following calculation:

\[ \int_{-1}^{1} \frac{1}{x^2} \, dx = \left[ -\frac{1}{x} \right]_{-1}^{1} = -2. \]

Her friend said: “But \( \frac{1}{x^2} \) is always positive, so its graph is above the \( x \)-axis, and the answer should be positive.”

Was either student correct? Give reasons for your answer.

2. (a) Use the Fundamental Theorem to find \( \int_{1}^{a} \frac{1}{x^2} \, dx \) (where \( a \) is any real number greater than 1).

(b) What is \( \lim_{a \to \infty} \int_{1}^{a} \frac{1}{x^2} \, dx \)?

(c) Draw a sketch of the function \( \frac{1}{x^2} \) and use it to interpret your answer to part (b) geometrically.

(d) Find a lower Riemann sum for the function \( \frac{1}{x^2} \) on the interval \([1, \infty)\), using sub-intervals of width 1.

(e) Use parts (b) and (d) to show that \( \sum_{n=1}^{\infty} \frac{1}{n^2} < 2. \)

(f) Calculate \( \sum_{n=1}^{7} \frac{1}{n^2} \).

(g) Deduce that \( 1.5 < \sum_{n=1}^{\infty} \frac{1}{n^2} < 2. \)

3. Use an appropriate integral, and Riemann sums, to find an estimate for

\[ 1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{100}. \]