This tutorial covers material from the lectures in Day 3.

1. Evaluate the following integrals:
   
   (a) \( \int_0^{5\pi} \cos \left( \frac{x}{10} \right) \, dx \).
   
   \( \text{Solution:} \quad 10. \)
   
   (b) \( \int_1^2 \sqrt{x-1} \, dx \).
   
   \( \text{Solution:} \quad \frac{2}{3}. \)
   
   (c) \( \int_0^1 x^\pi \, dx \).
   
   \( \text{Solution:} \quad \frac{1}{\pi + 1}. \)

2. Sketch the region bounded by the curves \( y = \sin x \) and \( y = \sin 2x \), and the straight lines \( x = 0 \) and \( x = \pi/2 \). Find the area of this region.

   \( \text{Solution:} \quad \) The curves intersect when \( \sin x = \sin 2x \), i.e., when \( \sin x = 2 \sin x \cos x \) or \( \sin x(1 - 2 \cos x) = 0 \). The roots occur when \( \sin x = 0 \) and \( \cos x = 1/2 \). For \( 0 \leq x \leq \pi/2 \), \( \sin x = 0 \) when \( x = 0 \) and \( \cos x = 1/2 \) when \( x = \pi/3 \).

   \[
   \int_0^{\pi/3} (\sin 2x - \sin x) \, dx + \int_{\pi/3}^{\pi/2} (\sin x - \sin 2x) \, dx
   \]
   
   \[
   = \left[ -\frac{\cos 2x}{2} + \cos x \right]_0^{\pi/3} + \left[ -\cos x + \frac{\cos 2x}{2} \right]_{\pi/3}^{\pi/2}
   \]
   
   \[
   = \frac{1}{4} + \frac{1}{2} + \frac{1}{2} - 1 + 0 - \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{1}{2}.
   \]
3. Suppose that a bagel cut horizontally in half has the shape given by rotating, about the \(y\) axis, the area bounded by the curve \(y = 3x - x^2 - 2\) and the \(x\)-axis.

(a) Sketch the curve \(y = 3x - x^2 - 2\).

**Solution:** The bagel has the following form:

(b) Consider a vertical strip under the graph at some point \(x_i\) (1 \(\leq \) \(x_i\) \(\leq \) 2), of width \(\Delta x\), which is rotated about the \(y\) axis to obtain a cylindrical shell. Sketch the shell.

**Solution:** The elementary shell is also shown in the diagram.

(c) Imagine the shell being cut vertically and opened out flat. Thus find the volume of the cylindrical shell.

**Solution:** The cylindrical shell has radius \(x\), and height \((3x - x^2 - 2)\). When we make the vertical cut and open it out, we obtain an object which approximates a rectangular slab of length \(2\pi x\) and height \((3x - x^2 - 2)\). The area of this slab is \(\Delta A = 2\pi x \times (3x - x^2 - 2) = 2\pi(3x^2 - x^3 - 2x)\), and the volume is \(\Delta V = \Delta A \times \Delta x = 2\pi(3x^2 - x^3 - 2x)\Delta x\).

(d) Write down the volume of the half bagel as a definite integral.

**Solution:** The volume of the half bagel, as a definite integral, is

\[
V = \int_{1}^{2} 2\pi(3x^2 - x^3 - 2x)dx .
\]

(e) Evaluate the definite integral to find the volume of the half bagel.

**Solution:**

\[
V = 2\pi \left[ x^3 - \frac{1}{4}x^4 - x^2 \right]_1^2 \\
= 2\pi \left( (8 - 4 - 4) - (1 - \frac{1}{4} - 1) \right) \\
= 2\pi \times \frac{1}{4} = \frac{\pi}{2}.
\]

4. Evaluate the following integrals by making a substitution.

(a) \(\int_{0}^{1} \frac{x^2}{\sqrt{2 + x^3}}dx\).

**Solution:** Use the substitution \(u = 2 + x^3\); then \(du = 3x^2dx\), \(u(0) = 2\), \(u(1) = 3\) and so

\[
\int_{0}^{1} \frac{x^2}{\sqrt{2 + x^3}}dx = \int_{2}^{3} \frac{1}{3}du = \left[ \frac{2}{3} \sqrt{u} \right]_{2}^{3} = \frac{2\sqrt{3}}{3} - \frac{2\sqrt{2}}{3}.
\]
(b) \[ \int_{0}^{\pi/2} \cos^3 x \, dx \] (Hint: First use the identity \( \cos^2 x = 1 - \sin^2 x \).)

**Solution:** Use the identity \( \cos^2 x = 1 - \sin^2 x \) and the substitution \( u = \sin x \); then \( du = \cos x \, dx \), \( u(0) = 0 \), \( u(\pi/2) = 1 \) and so

\[
\int_{0}^{\pi/2} \cos^3 x \, dx = \int_{0}^{\pi/2} (1 - \sin^2 x) \cos x \, dx = \int_{0}^{1} (1 - u^2) \, du = \left[ u - \frac{u^3}{3} \right]_{0}^{1} = \frac{2}{3}.
\]