This tutorial covers material from the lectures in Day 4.

1. Evaluate the following integrals by using integration by parts.

(a) \( \int_0^{1/2} x e^{2x} \, dx \).

Solution: Choose \( u = x \) and \( dv = e^{2x} \, dx \). Then \( du = dx \) and \( v = \frac{1}{2} e^{2x} \, dx \).

\[
\int_0^{1/2} x e^{2x} \, dx = \left[ \frac{1}{2} e^{2x} \right]_0^{1/2} - \int_0^{1/2} \frac{1}{2} e^{2x} \, dx = \left[ \frac{1}{2} e^{2x} \right]_0^{1/2} - \left[ \frac{1}{4} e^{2x} \right]_0^{1/2} = \frac{1}{4}.
\]

(b) \( \int_0^{\pi/4} \theta \sin 4\theta \, d\theta \).

Solution: Choose \( u = \theta \) and \( dv = \sin 4\theta \, d\theta \). Then \( du = d\theta \) and \( v = -\frac{1}{4} \cos 4\theta \). So,

\[
\int_0^{\pi/4} \theta \sin 4\theta \, d\theta = \left[ \theta \left( -\frac{1}{4} \cos 4\theta \right) \right]_0^{\pi/4} - \int_0^{\pi/4} \left( -\frac{1}{4} \cos 4\theta \right) d\theta = \left[ \theta \left( -\frac{1}{4} \cos 4\theta \right) \right]_0^{\pi/4} + \left[ \frac{1}{16} \sin 4\theta \right]_0^{\pi/4} = \frac{\pi}{16}.
\]

(c) \( \int_1^2 t^2 \ln t \, dt \).

Solution:

\[
\int_1^2 t^2 \ln t \, dt = \left[ \frac{1}{3} t^3 \ln t \right]_1^2 - \int_1^2 \frac{1}{3} t^3 \frac{1}{t} \, dt = \left[ \frac{1}{3} t^3 \ln t \right]_1^2 - \left[ \frac{t^3}{9} \right]_1^2 = \frac{8}{3} \ln 2 - \frac{7}{9}.
\]

2. Define \( \text{Si}(x) \) as \( \text{Si}(x) = \int_0^x f(t) \, dt \), where \( f(t) = \begin{cases} \sin t & t \neq 0 \\ t & t = 0 \end{cases} \).

This function is called the \textit{sine-integral}, and is useful in optics.
This is the graph of $f(t)$.

(a) What is $S_i'(x)$?

**Solution:** By the Fundamental Theorem of Calculus Part I, $S_i'(x) = \frac{\sin x}{x}$ for $x \neq 0$ and $S_i'(x) = 1$ when $x = 0$.

(b) What is the value of $S_i(0)$?

**Solution:** $S_i(0) = \int_0^0 f(t) \, dt = 0$.

(c) For $0 \leq x \leq 3\pi$, use the graph of $f(t)$ to determine the values of $x$ for which $S_i(x)$ is increasing, and the values of $x$ for which it is decreasing.

**Solution:** From (a), we have $S_i'(x) = (\sin x)/x$. Now $S_i(x)$ is increasing when $S_i'(x) > 0$, and decreasing when $S_i'(x) < 0$, so we need to determine the values of $x$ for which $(\sin x)/x$ is positive, and those for which it is negative. Looking at the graph provided, or noting that for $x > 0$ we have $(\sin x)/x > 0$ precisely when $\sin x > 0$ and $(\sin x)/x < 0$ precisely when $\sin x < 0$, we see that

$$(\sin x)/x > 0, \quad \text{for } 0 < x < \pi \text{ and } 2\pi < x < 3\pi$$

$$(\sin x)/x < 0, \quad \text{for } \pi < x < 2\pi.$$ 

So $S_i(x)$ is increasing for $0 < x < \pi$ and $2\pi < x < 3\pi$, and decreasing for $\pi < x < 2\pi$.

(d) For which values of $x$ between 0 and $3\pi$ does $S_i(x)$ have stationary points?

**Solution:** $S_i(x)$ has stationary points when $S_i'(x) = 0$. Looking at the graph again, or noting that $(\sin x)/x = 0$ precisely when $\sin x = 0$ (except at $x = 0$), we find the stationary points of $S_i(x)$ to be at $x = \pi, 2\pi$, and $3\pi$.

(e) Use the graph of $f(t)$ to estimate $S_i(\pi)$, $S_i(2\pi)$ and $S_i(3\pi)$.

**Solution:** Add up the squares under the graph of $(\sin x)/x$ to estimate $S_i(x)$, remembering that the area of each square is $0.2 \times 0.2 = 0.04$ and that the area under the $x$ axis is given a negative sign. This gives $S_i(\pi) \approx 1.9$, $S_i(2\pi) \approx 1.4$, $S_i(3\pi) \approx 1.7$. Note that these are approximate, and you may have obtained slightly different values.

(f) Use the graph of $f(t)$ to determine values of $x$ at which $S_i$ has points of inflection.

**Solution:** Since $(\sin x)/x$ is the derivative of $S_i$, the points of inflection of $S_i$ occur when $(\sin x)/x$ has stationary points. That is, when $x \approx 0, 4.5, 7.7$.\hfill 2
(g) Sketch the graph of $S_i$ for $0 \leq x \leq 3\pi$. Indicate clearly the vertical and horizontal scales on your graph. Be as accurate as possible. You may wish to use the grid overleaf as a guide.

Solution: The sketch of the function is

![Graph of $S_i$](image)

3. Establish the following reduction formula. (Hint: Write the integrand as $u(x)v'(x)$ where $u = \cos^{n-1}x$.)

$$
\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx .
$$

Use this formula to find $\int \cos^2 x \, dx$ and $\int \cos^4 x \, dx$.

Solution: Putting $u = \cos^{n-1} x$ and $v' = \cos x$ so that $v = \sin x$, we can integrate by parts:

$$
I_n = \int \cos^n x \, dx = \int \cos^{n-1} x \frac{d}{dx}(\sin x) \, dx
= \cos^{n-1} x \sin x - \int \sin x \frac{d}{dx}(\cos^{n-1} x) \, dx
= \cos^{n-1} x \sin x - \int \sin x(n - 1) \cos^{n-2} x(- \sin x) \, dx
= \cos^{n-1} x \sin x + (n - 1) \int \sin^2 x \cos^{n-2} x \, dx
= \cos^{n-1} x \sin x + (n - 1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx
= \cos^{n-1} x \sin x + (n - 1)I_{n-2} - (n - 1)I_n .
$$

Rearranging,

$$
nI_n = \cos^{n-1} x \sin x + (n - 1)I_{n-2} \quad \text{or} \quad I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n}I_{n-2} .
$$

Now $I_0 = \int \cos^0 x \, dx = \int dx = x + C$, so

$$
I_2 = \frac{1}{2} \cos x \sin x + \frac{1}{2}I_0 = \frac{1}{2} \cos x \sin x + \frac{1}{2}x + C_1
$$

and

$$
I_4 = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4}I_2 = \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8}x + C_2 .
$$
4. \(a\) Let \(I_n = \int x(\ln x)^n \, dx\). Use integration by parts to establish the reduction formula
\[
I_n = \frac{1}{2} x^2(\ln x)^n - \frac{n}{2} I_{n-1}.
\]

**Solution:** Since we cannot easily integrate \((\ln x)^n\), we choose \(u = (\ln x)^n\) and \(v' = x\), i.e. \(v = \frac{x^2}{2}\). Then
\[
\int x(\ln x)^n \, dx = \int (\ln x)^n \frac{d}{dx}\left(\frac{x^2}{2}\right) \, dx
= \frac{1}{2} x^2 (\ln x)^n - \int \frac{x^2}{2} \frac{d}{dx}(\ln x)^n \, dx
= \frac{1}{2} x^2 (\ln x)^n - \int \frac{x^2}{2} n(\ln x)^{n-1} \frac{1}{x} \, dx
= \frac{1}{2} x^2 (\ln x)^n - \frac{n}{2} \int x(\ln x)^{n-1} \, dx
= \frac{1}{2} x^2 (\ln x)^n - \frac{n}{2} I_{n-1}.
\]

\(b\) Starting with \(I_0 = \int x \, dx = \frac{1}{2} x^2 + C\), use the reduction formula from part \(a\) to find \(I_2\).

**Solution:** Starting with \(I_0 = \frac{1}{2} x^2 + C\),
\[
I_1 = \frac{1}{2} x^2 (\ln x) - \frac{1}{2} I_0 = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C_1,
\]
where \(C_1\) is a constant. Applying the formula again,
\[
I_2 = \frac{1}{2} x^2 (\ln x)^2 - \frac{2}{2} I_1 = \frac{1}{2} x^2 (\ln x)^2 - \frac{1}{2} x^2 \ln x + \frac{1}{4} x^2 + C_2.
\]