This tutorial covers material from the lectures in Day 5.

1. Simplify the expression $e^{x+\ln x}$.

   **Solution:** $e^{x+\ln x} = e^x e^{\ln x} = xe^x$.

2. Find $dy/dx$ for each of the following:

   (a) $y = \ln \left( \frac{1+x}{1-x} \right)$

      **Solution:** $\frac{2}{1-x^2}$

   (b) $y = \log_2 x$

      **Solution:** $\frac{1}{x \ln 2}$

   (c) $y = 3 \log_2(x^2)$

      **Solution:** $3 \log_2(x^2) = 6 \log_2(x) = \frac{6 \ln x}{\ln 2}$, so $\frac{dy}{dx} = \frac{6}{x \ln 2}$.

   (d) $y = x^x$

      **Solution:** $\frac{dy}{dx} = x^x \ln x$.

   (e) $y = \frac{x^2 \sqrt{7x-4}}{(1+x^2)^4}$

      **Solution:** $y = 2 \ln x + \frac{1}{3} \ln(7x - 4) - 4 \ln(1 + x^2)$.

      Differentiating both sides with respect to $x$, we have

      \[
      \frac{1}{y} \frac{dy}{dx} = \frac{2}{x} + \frac{1}{3} \times \frac{7}{7x - 4} - 4 \times \frac{2x}{1 + x^2}.
      \]

      Therefore, $\frac{dy}{dx} = \frac{x^2 \sqrt{7x-4}}{(1+x^2)^4} \left( \frac{2}{x} + \frac{7}{3(7x-4)} - \frac{8x}{1+x^2} \right)$. 

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3. (a) Use l’Hôpital’s rule to find \( \lim_{x \to 0} \ln \left( (1 + x)^{1/x} \right) \).

**Solution:** \( \lim_{x \to 0} \ln \left( (1 + x)^{1/x} \right) = \lim_{x \to 0} \frac{\ln(1 + x)}{x} = \lim_{x \to 0} \frac{1}{1 + x} = 1 \), where l’Hôpital’s rule has been used in the second step.

(b) Hence find \( \lim_{x \to 0} (1 + x)^{1/x} \).

[Note: \( \lim_{x \to 0} \ln \left( (1 + x)^{1/x} \right) = \ln \left( \lim_{x \to 0} (1 + x)^{1/x} \right) \). Can you say why?]  

**Solution:** Since \( \ln x \) is continuous,

\[
\lim_{x \to 0} \ln \left( (1 + x)^{1/x} \right) = \ln \left( \lim_{x \to 0} (1 + x)^{1/x} \right) = 1.
\]

Therefore, \( \lim_{x \to 0} (1 + x)^{1/x} = e \).

4. (a) For which positive real numbers \( x \) is it true that \( \sqrt{x} > x/2 \)?

**Solution:** If \( \sqrt{x} > x/2 \) then \( 4x > x^2 \) or \( (4 - x)x > 0 \) which requires \( 0 < x < 4 \).

(b) Without using your calculator, and assuming that \( \pi \approx 3 \), determine which of the following is bigger: \((\sqrt{\pi})^\pi\) or \(\pi^{\sqrt{\pi}}\).

[Hint: The exponential function is always increasing, so if \( a > b \), then \( e^a > e^b \).]

**Solution:** \((\sqrt{\pi})^\pi = \exp(\pi \ln \sqrt{\pi}) = \exp(\frac{\pi}{2} \ln \pi)\), and \(\pi^{\sqrt{\pi}} = \exp(\sqrt{\pi} \ln \pi)\).

Now, \( 0 < \pi < 4 \), and so \( \frac{\pi}{2} < \sqrt{\pi} \) (by part (a)).

Therefore \( \frac{\pi}{2} \ln \pi < \sqrt{\pi} \ln \pi \) (\( \ln \pi > 0 \)), and \( \exp \left( \frac{\pi}{2} \ln \pi \right) < \exp(\sqrt{\pi} \ln \pi) \).

That is, \((\sqrt{\pi})^\pi < \pi^{\sqrt{\pi}}\).