This tutorial covers material from the lectures in Day 10.

1. Classify and solve the differential equation \((1 + x^2) \frac{dy}{dx} = 1 - 2xy\).

   **Solution:** Since this equation is linear it should first be rearranged in standard form:

   \[
   (1 + x^2) \frac{dy}{dx} + 2xy = 1 \\
   \frac{dy}{dx} + \frac{2x}{1 + x^2}y = \frac{1}{1 + x^2}.
   \]

   Since \(p(x) = \frac{2x}{1 + x^2}\), the integrating factor is

   \[
   \exp \left( \int \frac{2x}{1 + x^2} \, dx \right) = \exp(\ln(1 + x^2)) = 1 + x^2,
   \]

   and we have

   \[
   \frac{d}{dx}((1 + x^2)y) = 1.
   \]

   Integrating produces

   \[
   (1 + x^2)y = \int dx = x + C, \quad \text{or} \quad y = \frac{x + C}{1 + x^2}.
   \]

2. Find whether each of the following first-order equations is separable, linear or neither. If the equation is separable or linear find its general solution.

   (a) \(\frac{dy}{dx} = \frac{x^2 - y}{x}\)

   **Solution:** The equation is not separable, however we can rearrange the equation as

   \[
   \frac{dy}{dx} + \frac{y}{x} = x,
   \]

   which we now recognize as first-order linear. The integrating factor is then

   \[
   r(x) = e^{\int (1/x) \, dx} = e^{\ln x} = x.
   \]

   Multiplying the equation by \(r(x)\), we obtain

   \[
   x \frac{dy}{dx} + y = x^2 \quad \text{or} \quad \frac{d}{dx}(xy) = x^2.
   \]
Integrating both sides with respect to \( x \), we get

\[ xy = \frac{x^3}{3} + C. \]

Thus we obtain the general solution \( y = \frac{x^2}{3} + \frac{C}{x} \).

(b) \( \frac{dy}{dx} = \frac{y - 1}{x(1 + x)} \)

**Solution:** The equation

\[ \frac{dy}{dx} = \frac{y - 1}{x(1 + x)} = (y - 1) \left( \frac{1}{x(1 + x)} \right) \]

is separable, so we separate and integrate:

\[ \int \frac{dy}{y - 1} = \int \frac{dx}{x(1 + x)} = \int \left( \frac{1}{x} - \frac{1}{1 + x} \right) dx \quad \text{(by partial fractions)}. \]

So

\[ \ln |y - 1| = \ln |x| - \ln |1 + x| + C = \ln \left| \frac{x}{1 + x} \right| + C. \]

Therefore,

\[ y - 1 = \frac{Ax}{1 + x}, \text{ where } A = \pm e^C, \]

and so the general solution is \( y = \frac{Ax}{1 + x} + 1. \)

(c) \( \frac{dy}{dx} = \frac{\sin x - y}{x - \cos y} \)

**Solution:** This equation is neither separable nor linear.

(d) \( \frac{dy}{dx} = \frac{y^2 - 1}{2(1 + x)y} \)

**Solution:** The equation

\[ \frac{dy}{dx} = \frac{y^2 - 1}{2(1 + x)y} = \left( \frac{y^2 - 1}{2y} \right) \left( \frac{1}{1 + x} \right) \]

is separable, so separate and integrate:

\[ \int \frac{2y}{y^2 - 1} dy = \int \frac{dx}{1 + x}. \]

This gives \( \ln |y^2 - 1| = \ln |1 + x| + C \), and so \( y^2 - 1 = A(1 + x) \), where \( A = \pm e^C \).

Thus we obtain the general solution \( y^2 = A(1 + x) + 1 = Ax + (A + 1). \)
3. Radiocarbon dating allows us to estimate the age of ancient objects. In living organisms, the ratio of radioactive carbon-14 to ordinary carbon-12 is constant. However, when the organism dies, carbon-14 is no longer absorbed (from the atmosphere or via feeding, for example), and so the amount present decreases through radioactive decay. By comparing the amount of carbon-14 present with the amount which would normally be present, we can determine the number of years since an organism died (or the age of objects such as clothing or paper, made from once-living material).

The half-life of carbon-14 is 5730 years. This means that the rate of decay is proportional to the amount \( y(t) \) of carbon-14 present,

\[
\frac{dy}{dt} = -ky \quad (k > 0),
\]

and that half of any given amount will disintegrate in 5730 years.

(a) Find the solution of this equation given that at time \( t = 0 \), the amount of carbon-14 present is \( y_0 \).

**Solution:** The equation is separable:

\[
\int \frac{1}{y} \, dy = \int (-k) \, dt,
\]

and thus we obtain \( \ln y = -kt + c \), i.e. \( y = Ae^{-kt} \) where \( A = e^c \). Since \( y(0) = y_0 \), we have \( y_0 = Ae^0 = A \). So \( A = y_0 \) and \( y = y_0 e^{-kt} \).

(b) Use the fact that half of this amount \( y_0 \) will disintegrate over 5730 years to find a numerical value for \( k \).

**Solution:** We are given that \( y(5730) = \frac{1}{2}y_0 \), and so \( \frac{1}{2}y_0 = y_0 e^{-5730k} \), which then gives \( k = (-1/5730) \ln(\frac{1}{2}) = 0.000121 \) (to 6 decimal places).

(c) A piece of woollen clothing is found to have only 77% of the amount of carbon-14 normally found in wool. Estimate the age of this piece of clothing.

**Solution:** If only 77% of the original amount of carbon-14 remains, we must have \( 0.77y_0 = y_0 e^{-0.000121t} \) and so \( t = (-1/0.000121) \ln(0.77) \approx 2160 \) years.
4. When people smoke, carbon monoxide is released into the air. In a room of volume 50 m$^3$, smokers introduce air containing 0.05 mg/m$^3$ of carbon monoxide at the rate of 0.002 m$^3$/min. Assume that the smoky air mixes immediately with the rest of the air, and that the mixture is pumped through an air purifier at a rate of 0.002 m$^3$/min. The purifier removes all the carbon monoxide from the air passing through it.

(a) Write a differential equation for $m(t)$, the mass of carbon monoxide in the room at time $t$, where $t$ is measured in minutes.

**Solution:** If $m(t)$ is the mass of carbon monoxide in the room at time $t$, then
\[
\frac{dm}{dt} = \text{mass rate in} - \text{mass rate out}.
\]
The smoky air is entering at the rate of 0.002 m$^3$/min and contains 0.05 mg/m$^3$ of CO. So the rate at which CO is introduced is $0.05 \times 0.002$ mg/min. At time $t$, the 50 m$^3$ room contains $m$ mg of CO, and so the concentration of CO is $\frac{m}{50}$ mg/m$^3$ and the rate at which it is removed is $0.002 \times \frac{m}{50}$ mg/min. Therefore
\[
\frac{dm}{dt} = 0.05 \times 0.002 - 0.002 \times \frac{m}{50} = 0.00002(5 - 2m).
\]

(b) Solve the differential equation, assuming that there was no carbon monoxide in the room initially.

**Solution:** This is a separable equation:
\[
\int \frac{dm}{5 - 2m} = 0.00002 \int dt
\]
\[-\frac{1}{2} \ln |5 - 2m| = 0.00002t + C
\]
\[5 - 2m = Ae^{-0.00004t} \quad (A = e^C).
\]
Since there is no carbon monoxide in the room initially, $m = 0$ when $t = 0$, and so $A = 5$. We therefore have
\[m = \frac{5}{2} \left(1 - e^{-0.00004t}\right).
\]

(c) What happens to the value of $m(t)$ in the long run?

**Solution:** In the long run $m \to \frac{5}{2}$. That is, in the 50 m$^3$ room, the concentration of CO is eventually $2.5/50$ or 0.05 mg/m$^3$.

(d) It is dangerous for people to be in the room if the mass of carbon monoxide per unit volume reaches 0.001 mg/m$^3$. How long does it take for this to happen?

**Solution:** We want to find the time at which the mass of carbon monoxide is $0.001 \times 50 = 0.05$ mg. So we have
\[0.05 = \frac{5}{2} \left(1 - e^{-0.00004t}\right),
\]
and hence $t = -\frac{\ln(0.98)}{0.00004} \approx 500$ min.