1. Find the particular solution of \( \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} - 8y = 0 \) which satisfies \( y(0) = 0 \) and \( y'(0) = 3 \).

**Solution:** The auxiliary equation is \( \lambda^2 + 2\lambda - 8 = 0 \) so that \( \lambda = 2 \) or \( \lambda = -4 \). Therefore, the general solution is \( y = Ae^{2t} + Be^{-4t} \).

If \( y(0) = 0 \), then \( A + B = 0 \) (and \( B = -A \)). Now, \( y'(t) = 2Ae^{2t} - 4Be^{-4t} \), so \( y'(0) = 3 = 2A - 4B = 6A \). Therefore, \( A = \frac{1}{2} \) and \( B = -\frac{1}{2} \), and the particular solution is \( y = \frac{1}{2}e^{2t} - \frac{1}{2}e^{-4t} \).

2. Find the general solution of \( \frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} + 5y = 0 \), expressing your answer in terms of real functions.

What is the particular solution satisfying \( y(0) = 1 \) and \( y(\pi/4) = 2 \)?

**Solution:** The auxiliary equation is \( \lambda^2 - 2\lambda + 5 = 0 \).

The roots of this equation are \( \lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i \). The general solution of the differential equation is therefore

\[
y = Ae^{(1+2i)t} + Be^{(1-2i)t} = e^t(Ae^{2t} + Be^{-2t}).
\]

In terms of real functions, the general solution is

\[
y = e^t(C \cos 2t + D \sin 2t).
\]

If \( y(0) = 1 \), then \( C = 1 \). If \( y(\pi/4) = 2 \), then \( 2 = e^{\pi/4}D \), and \( D = 2e^{-\pi/4} \). The required particular solution is therefore \( y = e^t(\cos 2t + 2e^{-\pi/4} \sin 2t) \).

3. Find the particular solution of \( \frac{d^2 x}{dt^2} + 2 \frac{dx}{dt} + x = 0 \) which satisfies \( x(0) = 1 \) and \( x'(0) = 2 \).

**Solution:** The auxiliary equation is \( \lambda^2 + 2\lambda + 1 = 0 \) and this has two equal roots \( \lambda = -1 \). Since the auxiliary equation has equal roots, the general solution to the differential equation is

\[
x = Ate^{-t} + Be^{-t}.
\]

When \( t = 0 \), \( x = B \), and so \( B = 1 \) and \( x = Ate^{-t} + e^{-t} \).

Now, \( x'(t) = A(-te^{-t} + e^{-t}) - e^{-t} \), so \( x'(0) = A - 1 = 2 \) and \( A = 3 \). The particular solution is therefore \( x = 3te^{-t} + e^{-t} \).
4. Two species struggling to compete against each other in the same environment have populations at time \( t \) of \( x(t) \) and \( y(t) \), satisfying the equations

\[
\begin{align*}
x'(t) &= 3x(t) - 4y(t), \\
y'(t) &= -2x(t) + y(t).
\end{align*}
\]

Find the second-order differential equation satisfied by \( x(t) \).
Hence find \( x(t) \) and \( y(t) \).

**Solution:** Differentiating \( x' = 3x - 4y \) gives \( x'' = 3x' - 4y' \). But \( y' = -2x + y \), so

\[
x'' = 3x' - 4(-2x + y) \\
= 3x' + 8x - 4y \\
= 3x' + 8x + (x' - 3x) \quad \text{(since } -4y = x' - 3x) \\
= 4x' + 5x.
\]

That is, \( x'' - 4x' - 5x = 0 \).

For \( x'' - 4x' - 5x = 0 \), the auxiliary equation is \( m^2 - 4m - 5 = 0 \) with roots \( m = 5 \) or \( m = -1 \).

So \( x = Ae^{5t} + Be^{-t} \). Hence \( x' = 5Ae^{5t} - Be^{-t} \), and from \( x' = 3x - 4y \) we obtain

\[
y = \frac{1}{4}(3x - x') \\
= \frac{1}{4}[3(Ae^{5t} + Be^{-t}) - (5Ae^{5t} - Be^{-t})] \\
= -\frac{1}{2}Ae^{5t} + Be^{-t}.
\]

5. Two species are in a predator-prey relationship. One species, which numbers \( Y \) individuals, eats the other, which numbers \( X \) individuals. Historically the numbers of these species have been constant with \( X = 3000 \) and \( Y = 1500 \). After a severe environmental disturbance the populations cease to be constant and start to change with time.

Let \( x(t) \) and \( y(t) \) be the difference between the historically constant population numbers and the new, changing population numbers \( X(t) \) and \( Y(t) \).

Then \( x(t) = X(t) - 3000 \) and \( y(t) = Y(t) - 1500 \) are the sizes of the perturbations from the historically steady states.

The sizes of the perturbations are described by the following:

\[
\begin{align*}
x'(t) &= 3x(t) - 2y(t) \\
y'(t) &= 4x(t) - y(t).
\end{align*}
\]

(a) Show that \( x''(t) - 2x'(t) + 5x(t) = 0 \).

**Solution:**

\[
\begin{align*}
x' &= 3x - 2y \\
x'' &= 3x' - 2y' \\
&= 3x' - 2(4x - y) \\
&= 3x' - 8x + 2y \\
&= 3x' - 8x + (3x - x') \\
&= 2x' - 5x.
\end{align*}
\]
That is, \( x'' - 2x' + 5x = 0. \)

(b) Find \( x(t) \) if \( x(0) = 100 \) and \( x'(0) = 100. \) (Take \( t = 0 \) to be the time at which monitoring of the population sizes begins.)

**Solution:** The auxiliary equation is \( m^2 - 2m + 5 = 0 \), with roots \( m = 1 + 2i \) and \( m = 1 - 2i. \) Therefore \( x = e^t (A \cos 2t + B \sin 2t). \)

If \( x = 100 \) when \( t = 0 \), then \( A = 100 \) and \( x = e^t (100 \cos 2t + B \sin 2t). \)

Hence, \( x' = e^t (-200 \sin 2t + 2B \cos 2t) + e^t (100 \cos 2t + B \sin 2t). \) When \( t = 0, \)

\( x' = 2B + 100 = 100, \) and so \( B = 0. \) Therefore \( x = 100e^t \cos 2t. \)

(c) Hence find \( y(t). \)

**Solution:**

\[
y(t) = \frac{1}{2} (3x - x') = \frac{1}{2} [300e^t \cos 2t - 100e^t \cos 2t + 200e^t \sin 2t] = 100e^t (\cos 2t + \sin 2t).
\]

(d) Sketch \( x(t) \) and \( y(t) \) and then \( X(t) \) and \( Y(t) \) as a function of \( t. \) What will happen to the original populations?

**Solution:**

The populations will oscillate with increasingly large swings. Eventually the swings will get so big that \( X(t) = 3000 + x(t) \) will be zero. (Once this happens the predictions of the model will no longer be valid.) The prey will then become extinct. The predator will then also become extinct unless it has an alternative source of food.