The University of Sydney

School of Mathematics and Statistics
Summer School 2012 - SS1003 Quiz 2

Name: SUGGESTED SOLUTIONS

SID

Signature:

This Question Sheet must be handed in intact

This is one of 3 quizzes to be conducted in the tutorials on Tuesday 24th January, Monday 6th February and Tuesday 14th February. The best two of the three possible quiz marks will contribute 25% to the assessment for the subject.

Approved non-programmable calculators may be used (but should not be needed).

Full marks will only be given if relevant working, reasoning and diagrams are included.

Please write in Ink.

Time Allowed :30 minutes -Total Marks : 25

Q1a (5 marks) Find the general solution of the differential equation

$$\frac{1}{x^2}\frac{dy}{dx} = \sin x^3$$

and then find the particular solution which satisfies $y(0) = 1$.

$$\frac{dy}{dx} = x^2 \sin x^3$$

$$y = \int x^2 \sin x^3 \, dx + C$$

$$\text{let } x^3 = u \Rightarrow u' = 3x^2.$$

$$y = \frac{1}{3}\int 3x^2 \sin x^3 \, dx + C$$

$$= \frac{1}{3}\int u' \sin u \, dx + C$$

$$= \frac{1}{3}\int \sin u \, du + C$$

$$= -\frac{1}{3}\cos u + C$$

$$y(x) = -\frac{1}{3}\cos x^3 + C$$

$$y(0) = 1 = C - \frac{1}{3} \Rightarrow C = \frac{4}{3}.$$

$$\boxed{y(x) = \frac{1}{3}\left(4 - \cos x^3\right)}$$

Q1b (5 marks) Find the general solution of the differential equation

$$\frac{dx}{dt} = t^2 e^{-t}$$

and then find the particular solution which satisfies $x(0) = 1$

$x = \int t^2 e^{-t} dt + C$

use IBP: let $u = t^2 : v' = e^{-t}$

then $u' = 2t : v = -e^{-t}$

$uv - \int u'v dx = -t^2 e^{-t} - \int (-e^{-t})(2t) dt$.

$= -t^2 e^{-t} + 2\int t e^{-t} dt$

use IBP again let $r = t : s' = e^{-t}$

$r' = 1 \quad s = -e^{-t}$

$rs - \int r's dt$

$\int t e^{-t} dt = -te^{-t} - \int (-e^{-t}) \cdot 1 dt$

$= -te^{-t} + \int e^{-t} dt$

$= -te^{-t} - e^{-t} + C$

so $x(t) = -t^2 e^{-t} + 2(-te^{-t} - e^{-t}) + C$

$= e^{-t}(-t^2 - 2t - 2) + C$

$x(0) = e^{-0}(\quad -2) + C = 1 \Rightarrow C = 3$

$\boxed{x(t) = 3 - e^{-t}(t^2 + 2t + 2)}$

Q2a (3 marks) Find $\frac{dy}{dx}$ if

$$y = (2x)^{x^3}$$

$$\ln y = x^3 \ln 2x$$

$$\frac{d}{dx}: \quad \frac{y'}{y} = 3x^2 \ln 2x + x^3 \times \frac{2}{2x}$$

$$\boxed{y' = (2x)^{x^3}\left(3x^2 \ln 2x + x^2\right)}$$

Q2b (3 marks) Find $\frac{dy}{dx}$ if

$$y = \log_3 \sin^2 x$$

$$y = \frac{\ln \sin^2 x}{\ln 3}$$

$$= \frac{2 \ln \sin x}{\ln 3}$$

$$y' = \frac{2}{\ln 3} \times \frac{1}{\sin x} \times \cos x$$

$$\boxed{y' = \frac{2 \cot x}{\ln 3}}$$

Q2c (3 marks) Find $\frac{dy}{dx}$ if

$$y = (\ln(\cos x))^x$$

take ln b.s.

$$\ln y = x \ln(\ln \cos x)$$

$\frac{d}{dx}$ b.s.

$$\frac{y'}{y} = \ln(\ln(\cos x)) + x \times \frac{1}{\ln(\cos x)} \times \frac{d \ln(\cos x)}{dx}$$

$$= \ln(\ln(\cos x)) \frac{+x}{\ln(\cos x)} \times \frac{1}{\cos x}(-\sin x)$$

So $\boxed{y' = (\ln(\cos x))^x \left(\ln(\ln(\cos x)) - \frac{x \tan x}{\ln(\cos x)} \right)}$

Q3a (3 marks) Find the general solution of the differential equation

$$\frac{dy}{dx} = \log_2 x$$

$$= \frac{\ln x}{\ln 2}$$

$$y = \frac{1}{\ln 2} \int \ln x \, dx$$

Use IBP let $u = \ln x$ $v' = 1 \Rightarrow u' = \frac{1}{x}$; $v = x$

$(uv - \int u'v \, dx.)$

$$y = \frac{1}{\ln 2}\left(x \ln x - \int \frac{1}{x} x \, dx\right)$$

$$\boxed{y(x) = \frac{1}{\ln 2}\left(x \ln x - x\right) + C.}$$

$$= x \log_2 x - \frac{x}{\ln 2} + C$$

Q3 continues next page

Q3b (2 marks) Find the particular solution which satisfies

$$y(8) = 24.$$

$$y(8) = 8\log_2 8 - \frac{8}{\ln 2} + C$$

$$= 24 - \frac{8}{\ln 2} + C = 24$$

$$\text{so} \qquad C = \frac{8}{\ln 2}$$

$$\boxed{y(x) = x\log_2 x + \frac{1}{\ln 2}(8 - x)}$$

Q3c (1 mark) Verify then that

$$y(4) = 8 + \frac{4}{\ln 2}$$

$$\text{so } y(4) = 4\log_2 4 + \frac{1}{\ln 2}(8-4)$$

$$= 4 \times 2 + \frac{4}{\ln 2}$$

$$\boxed{y(4) = 8 + \frac{4}{\ln 2}}$$

End of Quiz Questions)