The University of Sydney
Math1003 Integral Calculus and Modelling

Summer School	**Exercises for Tute 5**	2007

Assumed Knowledge: Sketching curves of simple functions. Integrals of simple functions such as x^n (including $1/x$), $\sin x$, $\cos x$, e^x.

Objectives:

(4a) To understand and be able to use integration by parts to evaluate definite integrals.

(4b) To understand that an indefinite integral is a function.

(4c) To understand that differentiation and (indefinite) integration are inverse processes when applied to functions.

(4d) To be able to sketch a function given its derivative.

(4e) To be able to derive a reduction formula for an integral.

Preparatory questions:

1. Find the indefinite integrals

(i) $\displaystyle\int \tan x\, dx$. Hint: Write $\tan x = \sin x/\cos x$ and choose a suitable substitution.

(ii) $\displaystyle\int x^2 e^x\, dx$.

Practice Questions:

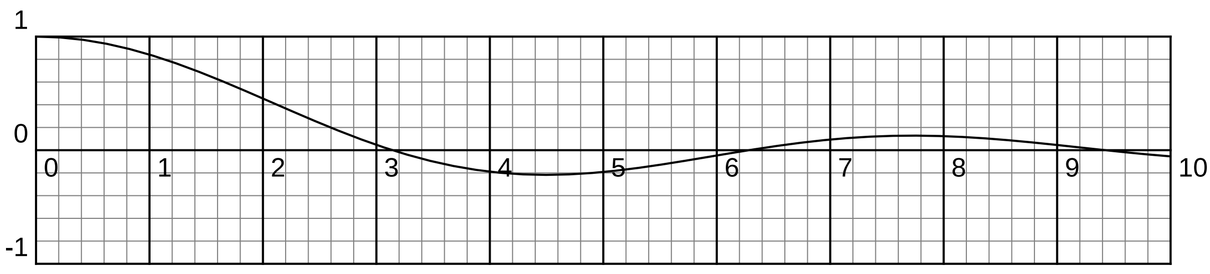
2. Evaluate the following integrals by using integration by parts.

(i) $\displaystyle\int_0^{1/2} xe^{2x}\, dx$. (ii) $\displaystyle\int_0^{\pi/4} \theta \sin 4\theta\, d\theta$. (iii) $\displaystyle\int_1^2 t^2 \ln t\, dt$.

3. Define $\mathrm{Si}(x)$ as $\mathrm{Si}(x) = \displaystyle\int_0^x f(t)\, dt$, where $f(t) = \begin{cases} \dfrac{\sin t}{t} & t \neq 0 \\ 1 & t = 0\,. \end{cases}$

This function is called the *sine-integral*, and is useful in optics.

This is the graph of $f(t)$.



(i) What is $\text{Si}\,'(x)$?. (ii) What is the value of $\text{Si}(0)$?

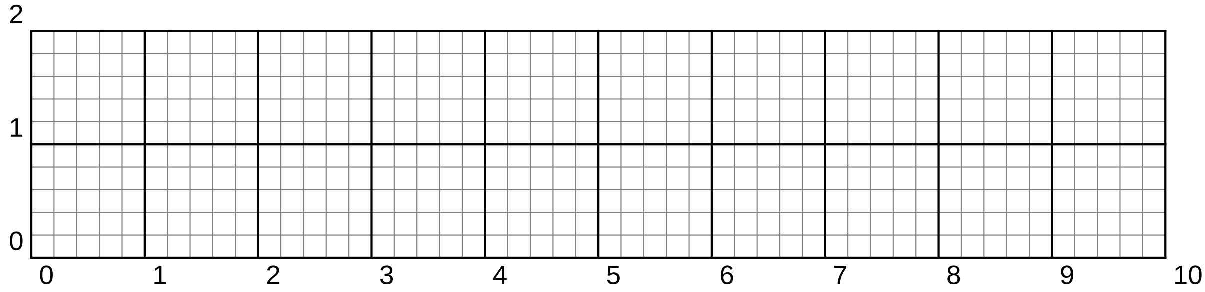
(iii) For $0 \le x \le 3\pi$, use the graph of $f(t)$ to determine the values of x for which $\text{Si}(x)$ is increasing, and the values of x for which it is decreasing.

(iv) For which values of x between 0 and 3π does $\text{Si}(x)$ have stationary points?

(v) Use the graph of $f(t)$ to estimate $\text{Si}(\pi)$, $\text{Si}(2\pi)$ and $\text{Si}(3\pi)$.

(vi) Use the graph of $f(t)$ to determine values of x at which Si has points of inflection.

(vii) Sketch the graph of Si for $0 \le x \le 3\pi$.



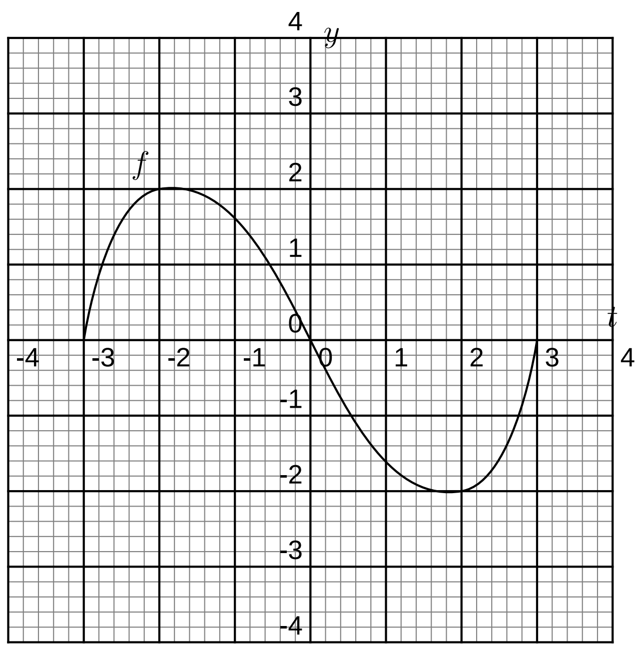
4. Establish the following reduction formula. [Hint: Write the integrand as $u(x)v'(x)$ where $u = \cos^{n-1} x$.]

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx\,.$$

Use this formula to find $\int \cos^2 x\,dx$ and $\int \cos^4 x\,dx$.

More Exercises

5. Let $g(x) = \int_{-3}^{x} f(t)\,dt$ where f is the *odd* function whose graph is shown.



(i) Evaluate $g(-3)$ and $g(3)$.

(ii) Estimate $g(-2)$, $g(-1)$ and $g(0)$.

(iii) On what interval is g increasing?

(iv) Where does g have a maximum value?

(v) Sketch a rough graph of g.

6. Evaluate the following integrals by using integration by parts.

(i) $\displaystyle\int_0^1 (2x+3)e^x\,dx.$ (ii) $\displaystyle\int_0^\pi \theta^2\cos 3\theta\,d\theta.$ (iii) $\displaystyle\int_{-\pi/4}^{\pi/4} t\sin t\cos t\,dt.$

Hint: First use an identity in (iii).

7. Let $I_n = \int x^n e^x dx$. Use integration by parts to establish the reduction formula

$$I_n = x^n e^x - nI_{n-1}\,.$$

Hence find $\int x^3 e^x dx$.

8. (i) Let $I_n = \int x(\ln x)^n dx$. Use integration by parts to establish the reduction formula

$$I_n = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}\,.$$

(ii) Starting with $I_0 = \int x\,dx = \frac{1}{2}x^2 + C$, use the reduction formula from part (i) to find I_2.

Answers to Selected Questions

1. (i) $-\ln|\cos x| + C.$ (ii) $(x^2 - 2x + 2)e^x + C.$

2. (i) $\dfrac{1}{4}.$ (ii) $\dfrac{\pi}{16}.$ (iii) $\dfrac{8}{3}\ln 2 - \dfrac{7}{9}.$

3. (iii) $\mathrm{Si}(x)$ is increasing for $0 < x < \pi$ and $2\pi < x < 3\pi$, and decreasing for $\pi < x < 2\pi$.

(iv) π, 2π and 3π

(v) $\mathrm{Si}(\pi) \approx 1.9$, $\mathrm{Si}(2\pi) \approx 1.4$, $\mathrm{Si}(3\pi) \approx 1.7$

(vi) 0, 4.5, 7.7

4. $I_2 = \dfrac{1}{2}\cos x\sin x + \dfrac{1}{2}x + C$

5. (i) $g(-3) = g(3) = 0$ (ii) $g(-2) \approx 1.4$, $g(-1) \approx 3.3$, $g(0) \approx 4.2$ (iii) $(-3, 0)$ (iv) $t = 0$

6. (i) $3e - 1.$ (ii) $-\dfrac{2\pi}{9}.$ (iii) $\dfrac{1}{4}.$

7. $I_3 = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

8. $I_2 = \dfrac{1}{2}x^2(\ln x)^2 - \dfrac{1}{2}x^2\ln x + \dfrac{1}{4}x^2 + C$