THE UNIVERSITY OF SYDNEY
MATH1003 INTEGRAL CALCULUS AND MODELLING

Semester 2 | **Exercises for Week 13** | 2010

Assumed Knowledge:

Objectives:

(12a) To be able to rewrite two coupled first-order differential equations as a single second-order differential equation.

(12b) To be able to sketch the solutions of second-order differential equations with constant coefficients.

Preparatory questions:

1. Two species, struggling to compete against each other in the same environment, have populations at time t of $x(t)$ and $y(t)$, satisfying the equations

$$x'(t) = 3x(t) - 4y(t)\,, \qquad y'(t) = -2x(t) + y(t)\,.$$

Find the second-order differential equation satisfied by $x(t)$.

Practice Questions:

2. Find $x(t)$ and $y(t)$ in Preparatory Question 1.

3. Two species are in a predator-prey relationship. Let the predator species number Y, and the prey species number X individuals. Historically the numbers of these species have been constant at $X = 3000$ and $Y = 1500$. After a severe environmental disturbance the populations cease to be constant and start to change with time.

For some time variable t let $x(t)$ and $y(t)$ be the difference between the historically constant population numbers and the new, changing population numbers $X(t)$ and $Y(t)$. Then $x(t) = X(t) - 3000$ and $y(t) = Y(t) - 1500$ are the sizes of the perturbations from the historically steady states.
Suppose the perturbations satisfy:

$$x'(t) = 3x(t) - 2y(t)$$
$$y'(t) = 4x(t) - y(t).$$

(i) Show that $x''(t) - 2x'(t) + 5x(t) = 0$.

(ii) Find $x(t)$ if $x(0) = 100$ and $x'(0) = 100$. (Take $t = 0$ to be the time at which monitoring of the population sizes begins.)

(iii) Hence find $y(t)$.

(iv) Sketch $x(t)$ and $y(t)$ and then $X(t)$ and $Y(t)$ as a function of t.
What does the model predict will happen to the original populations?

More Exercises

4. Find the general solution of the following system of equations:

$$\frac{dx}{dt} = x + y, \qquad \frac{dy}{dt} = 3x - y\,.$$

5. Find the general solution of the pair of differential equations

$$\frac{dx}{dt} = 5x - 3y\,, \qquad \frac{dy}{dt} = 2y\,,$$

by first solving the second equation and then substituting into the first. (There are two equations, so you should have *two* arbitrary constants of integration at the end.) Find the particular solution satisfying the initial conditions $x = 1$, $y = 2$ when $t = 0$.

6. (*i*) For each of the following systems of differential equations for $x(t)$ and $y(t)$, find an equivalent second-order differential equation. (You do not need to find any solutions.)

(a) $x' = y + 4x, \qquad y' = 6x.$

(b) $x' = y, \qquad y' = x + 5y.$

(*ii*) Eliminate $y(t)$ from the following system to obtain a nonlinear second-order equation for $x(t)$. (You do not need to find any solutions.)

$$x' = y, \qquad y' = x - xy + 3t$$

7. Find whether each of the following first-order equations is separable, linear or neither. If the equation is separable or linear find its general solution.

(*i*) $\dfrac{dy}{dx} = \dfrac{x^2 - y}{x}$ (*ii*) $\dfrac{dy}{dx} = \dfrac{y - 1}{x(1 + x)}$

(*iii*) $\dfrac{dy}{dx} = \dfrac{\sin x - y}{x - \cos y}$ (*iv*) $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{2(1 + x)y}$

Answers to Selected Questions

1. $x'' - 4x' - 5x = 0.$

2. $x = Ae^{5t} + Be^{-t}$, $y = -\frac{1}{2}Ae^{5t} + Be^{-t}$.

3. (*ii*) $x = 100e^t \cos 2t$, (*iii*) $y = 100e^t(\cos 2t + \sin 2t)$.

4. $x = Ae^{2t} + Be^{-2t}$, $y = Ae^{2t} - 3Be^{-2t}$.

5. General sol'n: $x = Ae^{2t} + Be^{5t}$, $y = Ae^{2t}$. Particular sol'n: $x = 2e^{2t} - e^{5t}$, $y = 2e^{2t}$.

6. (*i*) (a) $x'' - 4x' - 6x = 0$, (b) $y'' - 5y' - y = 0$.

(*ii*) $x'' + xx' - x = 3t$.

7. (*i*) Linear, $y = \dfrac{x^2}{3} + \dfrac{C}{x}$.

(*ii*) Separable, $y = 1 + \dfrac{Ax}{1 + x}$. Also linear.

(*iii*) Neither separable nor linear.

(*iv*) Separable, $y^2 = 1 + A(1 + x)$. Also linear (for x a function of y).