Extended Answer Section

There are three questions in this section, each with a number of parts. Write your answers in the space provided below each part. If you need more space there are extra pages at the end of the examination paper.

1. A particle moves in the xy-plane so that each second it moves either one unit in the positive x-direction or one unit in the positive y-direction. The initial position of the particle is at the origin.

(a) What is the number of possible trajectories of the particle during the first 10 seconds?

(2 marks)

It has 2 choices at \( t = 0 \) \( \left[ \begin{array}{ll}
1 & 1
\end{array} \right] \)

\( t = 1 \) 10 times

\( \ldots \)

\( t = 9 \)

So to choose

Hence the number is \( 2^{10} = 1024 \).

(b) What is the number of possible trajectories of the particle from the origin to the point (4,6)?

(2 marks)

It must choose x-direction 4 times of the 10.

So there are \( \binom{10}{4} = 210 \) possibilities.
(c) What is the number of possible trajectories of the particle from the origin to the point (12, 6) without two consecutive moves in the y-direction? (2 marks)

The 6 moves in the y-direction can be chosen at 13 possible x-coordinates 0, 1, 2, ..., 12.

Hence, the number is \( \binom{13}{6} = 1716 \).

(d) What is the number of possible trajectories of the particle in the first 7 seconds without two consecutive moves in the y-direction? (2 marks)

The possible endpoints of the trajectories are (7, 0), (6, 1), (5, 2), (4, 3) and (3, 4). So the number is

\[
\binom{8}{0} + \binom{7}{1} + \binom{6}{2} + \binom{5}{3} + \binom{4}{4} = 1 + 7 + 15 + 10 + 1 = 34.
\]

(e) What is the number of possible trajectories of the particle in the first 7 seconds with at most five consecutive moves in any direction? (2 marks)

The total number of trajectories is \( \binom{7}{0} = 128 \).

There are 2 trajectories with 7 consecutive moves, and there are 4 trajectories with 6 consecutive moves.

Hence, the number is \( 128 - 2 - 4 = 122 \).
2. The following switching circuit represents a Boolean function \( f \) in three variables \( x, y, z \):

(a) Use the switching circuit to write down a Boolean expression representing the function \( f \). (2 marks)

\[ f = z' (x \lor y z) \lor y x z'. \]

(b) Complete the table of values for the function \( f \). (2 marks)
(c) Write down the Boolean expression for $f$ in disjunctive normal form. (2 marks)

$$f = x'y'z' \lor x'yz \lor x'y'z \lor xyz$$

(d) Apply the Karnaugh map method to find a simpler Boolean expression for $f$. (2 marks)

$$f = xy \lor x'z$$

(e) Draw a simpler switching circuit representing the function $f$. (2 marks)
3. A sequence \( (x_n \mid n \geq 0) \) satisfies the recurrence relation \( x_n = 3x_{n-2} + 2x_{n-3} \).

(a) Write down the corresponding characteristic equation and find its roots. (3 marks)

\[
\lambda^3 - 3\lambda^2 - 2 = 0
\]

\[
(\lambda - 2)(\lambda^2 + 2\lambda + 1) = 0
\]

\[
(\lambda - 2)(\lambda + 1)^2 = 0
\]

Hence the roots are

\[
\lambda_1 = 2, \quad \lambda_2 = \lambda_3 = -1.
\]

(b) Write down the general solution of the recurrence relation. (2 marks)

\[
x_n = (A + Bn)(-1)^n + C \cdot 2^n.
\]
(c) Find the particular solution satisfying the initial conditions $x_0 = 1$, $x_1 = 3$ and $x_2 = 2$. 

\[
\begin{align*}
A + C &= 1 \\
-A - B + 2C &= 3 \\
A + 2B + 4C &= 2
\end{align*}
\Rightarrow -A + 8C = 8
\]

\Rightarrow C = 1, \quad A = 0, \quad B = -1.

The particular solution is

\[x_n = \frac{n}{2} - (-1)^n n.
\]

(d) Using your answer to the previous part or otherwise write down a closed form of the generating function of the sequence $(x_n)$. 

\[
G(z) = \sum_{n=0}^{\infty} x_n z^n = \sum_{n=0}^{\infty} \frac{n}{2} z^n - \sum_{n=0}^{\infty} (-1)^n n z^n
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{n}{2} z^n + z \sum_{n=1}^{\infty} n(-z)^{n-1} \right) = \frac{1}{1-2z} + \frac{z}{(1+z)^2}
\]

\[
= \frac{1+3z-2z^2}{1-3z^2-2z^3}
\]

You may use next pages for your answers.