Extended Answer Section

There are three questions in this section, each with a number of parts. Write your answers in the space provided below each part. If you need more space there are extra pages at the end of the examination paper.

1. There are twelve movies playing at the cinema, and Andrew, having just finished his final exams, plans on going to see three this weekend.
   (a) Andrew very reasonably insists that the three movies he goes to see are all different.
      (i) How many possibilities are there if he doesn’t care in which order he sees the movies?
      (ii) How many possibilities are there if he does care about order?
           Express both of your answers as simple integers.

(b) On Friday Andrew plans on seeing two different movies back-to-back. If movies 1,...,7 are playing at 6pm, and movies 6,...,12 are playing at 8pm, how many pairs of movies can Andrew see, if he doesn’t care in which order he sees the movies? Express your answer as a simple integer.
(c) On Saturday Bob, Cathy, Dan and Edith join Andrew at the movies. The five friends plan to sit next to each other in a row. How many ways they choose to sit together if Bob and Cathy insist on not sitting next to each other? Express your answer as a simple integer.

(d) After the movie the five friends go to dinner at a restaurant with circular tables. In how many ways can they sit around a table, if only their relative positions matter, and Bob and Cathy still insist on not sitting next to each other? Express your answer as a simple integer.
2. (a) Prove by induction that \[ \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}. \]

(b) Let \(x, y, z\) be propositions. Construct the truth table for the proposition \[ f = (x \land (\sim y \Rightarrow z)) \Rightarrow \sim x. \]
(c) Use the rules of boolean algebra to show that the boolean expressions
\((x \lor y)(xz \lor xz') \lor xy \lor y\) and \(x \lor y\) are equivalent.

(d) Given 10 different apples and 4 different boxes, in how many ways can you arrange
the apples in the boxes so that each box contains at least one apple? Explain your
answer but you don’t have to express it as a simple integer; it may involve binomial
coefficients etc.
3. (a) Consider the recurrence relation \( a_n = 5a_{n-1} - 7a_{n-2} + 3a_{n-3} \).

\((i)\) Find the corresponding characteristic equation and compute its roots.

\((ii)\) Given initial conditions \( a_0 = 2, a_1 = 5 \) and \( a_2 = 12 \) find a closed formula for \( a_n \) for \( n \geq 0 \).
(iii) Find a closed form for the generating function $F(x) = \sum_{n=0}^{\infty} a_n x^n$. 
(b) Let $A$ be the set of paths from $(0,0)$ to $(n,n)$ such that each step is one unit up or one unit right. Let $B = C_{2n,n}$, the set of subsets of $\{1,\ldots,2n\}$ of cardinality $n$.

(i) Construct a bijection $f : A \rightarrow B$.

(ii) Prove that the function constructed in (i) is a bijection.
You may use next pages for your answers
This page may be used if you need more space for your answers