1. Consider the following argument:

“If you are a bird then you can sing. If you take singing lessons then you can sing. Therefore, if you take singing lessons then you are a bird.”

Let

\[ p = \text{“you are a bird”}, \]
\[ q = \text{“you take singing lessons”}, \]
\[ r = \text{“you can sing”}. \]

(a) [1 mark] Write down the above argument symbolically as a single compound proposition.

**Solution:** The above argument can be written as

\[ [(p \Rightarrow r) \land (q \Rightarrow r)] \Rightarrow (q \Rightarrow p). \]

(b) [2 marks] Construct a truth table for the compound proposition you wrote down in part (a).

**Solution:** Write \( s = (p \Rightarrow r) \land (q \Rightarrow r) \). Then a truth table for the proposition in part (a) is:

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(c) [1 mark] Is the above argument valid? Explain why.

**Solution:** The argument is invalid since \( [(p \Rightarrow r) \land (q \Rightarrow r)] \Rightarrow (q \Rightarrow p) \) is not a tautology. In particular, the case where you take singing lessons and can sing but are not a bird is consistent with the conditions, but not with the conclusion.

2. [4 marks] Prove

\[ 1^2 + 2^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \]

for all \( n \geq 0 \) using mathematical induction.

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1 http://www.maths.usyd.edu.au/u/UG/SS/SS1004/
**Solution:** Let $P(n)$ be the given statement. For $n = 0$, the statement $P(0)$ is

$$0 = \frac{0(0+1)(2 \times 0 + 1)}{6}$$

which is true.

Now suppose that $P(n)$ is true, i.e. that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. To prove $P(n+1)$, consider:

$$1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

(by the inductive hypothesis)

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6}$$

$$= \frac{(n+1)(2n^2 + 7n + 6)}{6}$$

$$= \frac{(n+1)((n+1) + 1)(2(n+1) + 1)}{6}.$$ 

Hence $P(n) \Rightarrow P(n+1)$ is true for $n \geq 0$. Therefore $P(n)$ is true for all $n \geq 0$ by induction.