This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 50% of the total examination; there are 25 questions; the questions are of equal value; all questions may be attempted.

Answers to the Multiple Choice questions must be entered on the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 50% of the total examination; there are 3 questions; the questions are of equal value; all questions may be attempted; working must be shown.

Approved non-programmable, non-graphics calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.
There are three questions in this section, each with a number of parts. Write your answers in the space provided below each part. If you need more space there are extra pages at the end of the examination paper.

1. The answers for all parts of this question should be given in a final number form.

A bag contains a large supply of candies of eight different colours. The candies of the same colour are indistinguishable.

(a) What is the number of possible collections of four candies (allowing repeated colours) taken from the bag? (2 marks)

The number of collections is
\[
\binom{4+8-1}{4} = \binom{11}{4} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4} = 330.
\]

(b) Suppose that 8 candies of different colours will be given to three children. Two of them, Kate and Natalie will get three each, while Bob will get two candies. What is the number of different ways to make this distribution? (2 marks)

The number is given by the multinomial coefficient
\[
\binom{8}{3,3,2} = \frac{8!}{3!3!2!} = 560.
\]
(c) What is the number of different ways to distribute 4 red and 4 green candies between these children in such a way that Kate and Natalie get three each, while Bob gets two candies? 

The number in question coincides with the number of ways to distribute 4 red candies in such a way that Kate and Natalie get at most three each, while Bob gets at most two. The possible distributions (K,N,B) are

\[(3, 1, 0), (2, 2, 0), (1, 3, 0), (3, 0, 1), (2, 1, 1), (1, 2, 1), (0, 3, 1), (2, 0, 2), (1, 1, 2), (0, 2, 2)\]

Hence the answer is 10.

(d) Suppose now that 12 red candies will be given to Kate, Natalie and Bob. What is the number of different ways to make this distribution so that each child gets a different number of candies and no one misses out? 

Suppose \(k\), \(n\) and \(b\) be the quantities of candies received by Kate, Natalie and Bob, respectively. Suppose first that \(k > n > b\). Then \(b \geq 1\) and \(k + n + b = 12\). The possible triples \((k, n, b)\) are

\[(9, 2, 1), (8, 3, 1), (7, 4, 1), (6, 5, 1), (7, 3, 2), (6, 4, 2), (5, 4, 3)\]

Since there are \(3! = 6\) permutations of the numbers \(k\), \(n\) and \(b\), the answer is \(7 \cdot 6 = 42\).
2. (a) Let \( p \) and \( q \) be propositions. Complete the truth table for the compound proposition \( f(p, q) \) given by \( (p \lor \sim q) \implies (p \iff q) \). (2 marks)

The table is found by

\[
\begin{array}{c|cc|c|c|c}
 p & q & p \lor \sim q & p \iff q & f(p, q) \\
\hline
T & T & T & T & T \\
T & F & T & F & F \\
F & T & F & F & T \\
F & F & T & T & T \\
\end{array}
\]

(b) In the notation of the previous part find two non-equivalent propositions \( h_1(q) \) and \( h_2(q) \) which depend only on \( q \), such that both propositions \( f(p, q) \lor h_1(q) \) and \( f(p, q) \lor h_2(q) \) are tautologies. (3 marks)

There are four possible truth tables for propositions \( h(q) \) depending only on \( q \) so that \( h(q) \) is equivalent to one of the four propositions \( q \land \sim q \) (a contradiction), \( q \lor \sim q \) (a tautology), \( q \lor \sim q \). By the truth table for \( f(p, q) \) found in part (a), the proposition \( f(p, q) \lor h(q) \) is a tautology if and only if the value of \( h(q) \) in the second line of the table is \( T \). There two non-equivalent propositions with this property: \( h_1(q) = q \lor \sim q \) and \( h_2(q) = \sim q \).
(c) Introduce the **Boolean addition** $\oplus$ as the operation $x \oplus y = xy' \lor x'y$. Complete the table of values for the Boolean function $x \oplus y$. 

\begin{center}
\begin{tabular}{ccc}
\hline
$x$ & $y$ & $x \oplus y$
\hline
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\hline
\end{tabular}
\end{center}

(d) Find values $a, b, c, d \in \{0, 1\}$ such that the Boolean expression $xy' \lor y$ is equivalent to $a \oplus bx \oplus cy \oplus dxy$. 

\begin{center}
The tables of values of the Boolean functions $xy' \lor y$ and $a \oplus bx \oplus cy \oplus dxy$ are given by
\begin{tabular}{ccc}
\hline
$x$ & $y$ & $xy' \lor y$ & $a \oplus bx \oplus cy \oplus dxy$
\hline
1 & 1 & 1 & $a \oplus b \oplus c \oplus d$
1 & 0 & 1 & $a \oplus b$
0 & 1 & 1 & $a \oplus c$
0 & 0 & 0 & $a$
\hline
\end{tabular}
\end{center}

Since the Boolean expressions are equivalent, the corresponding Boolean functions must be equal. Therefore, $a = 0$, $b = 1$, $c = 1$ and $d = 1$ so that the equivalent Boolean expression is $x \oplus y \oplus xy$. 

3. (a) A sequence $a_n$ is defined by the recurrence relation $a_n = -a_{n-1} + 3^n$, $n \geq 1$, with $a_0 = 2$.

(i) Find a closed form of the generating function $G(z)$ of the sequence $a_n$. (3 marks)

Let 

$$G(z) = a_0 + a_1z + a_2z^2 + \ldots$$

Then 

$$G(z) + zG(z) = a_0 + (a_1 + a_0)z + (a_2 + a_1)z^2 + \ldots$$

$$= 2 + 3z + 3^2z^2 + \cdots = 1 + \frac{1}{1 - 3z} = \frac{2 - 3z}{1 - 3z}.$$ 

Hence, 

$$G(z) = \frac{2 - 3z}{(1 + z)(1 - 3z)}.$$ 

(ii) Using your answer to the previous part or otherwise find an explicit formula for $a_n$. (3 marks)

Use partial fraction decomposition for $G(z)$, 

$$\frac{2 - 3z}{(1 + z)(1 - 3z)} = A \frac{1}{1 + z} + B \frac{1}{1 - 3z}.$$ 

The coefficients $A$ and $B$ are found from $A(1 - 3z) + B(1 + z) = 2 - 3z$ so that $A = 5/4$ and $B = 3/4$. Hence, 

$$a_n = \frac{5}{4} (-1)^n + \frac{3}{4} 3^n.$$
(b) Use the characteristic equation to find the general solution of the recurrence relation
\[ b_n + 8b_{n-1} + 16b_{n-2} = 0, \quad n \geq 2. \]

The characteristic equation takes the form \( \lambda^2 + 8\lambda + 16 = 0 \), that is, \((\lambda + 4)^2 = 0\). The root \( \lambda = -4 \) has multiplicity 2, hence the general solution is \( b_n = (A + Bn)(-4)^n \), there \( A \) and \( B \) are arbitrary constants.

(c) Which of the sequences \( b_n \) satisfying the recurrence relation of the previous part also satisfy the relation \( b_n + 4b_{n-1} = (-4)^n, \quad n \geq 1? \)

Substituting \( b_n = (A + Bn)(-4)^n \) into the relation, we get
\[
(A + Bn)(-4)^n + 4(A + Bn - B)(-4)^{n-1} = (-4)^n.
\]
This simplifies to \( B(-4)^n = (-4)^n \) for \( n \geq 1 \). Hence \( B = 1 \) and the sequences satisfying the recurrence relation are \( b_n = (A + n)(-4)^n \).