Extended Answer Section

There are three questions in this section, each with a number of parts. Write your answers in the space provided below each part. If you need more space there are extra pages at the end of the examination paper.

1. Twenty books are to be placed in a bookcase with five shelves. The order of books on each shelf is irrelevant. Suppose that the books are all different.

   (a) What is the number of ways to place the books so that each shelf contains four books? (2 marks)

   The number of ways is given by the multinomial coefficient
   \[
   \binom{20}{4, 4, 4, 4, 4} = \frac{20!}{(4!)^5}
   \]
   which can also be found as the product
   \[
   \binom{20}{4}\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}.
   \]

   (b) What is the number of ways to place the books so that each shelf contains a different number of books and there are at least two books on each shelf? (2 marks)

   The numbers of books on the shelves should be equal to 2, 3, 4, 5, 6. Since there are 5! permutations of these numbers, the total number of ways to place the books equals 5! times the multinomial coefficient
   \[
   5!\binom{20}{2, 3, 4, 5, 6} = \frac{20!}{2!3!4!6!}.
   \]
Suppose now that the twenty books are an edition of a two-volume textbook with ten copies of each volume.

(c) What is the number of ways to place the books on the five shelves so that each shelf contains at least one copy of each volume? (3 marks)

Place one copy of the first volume on each shelf. Then we are left with 5 copies to be distributed amongst the five shelves. The number of such distributions is found as the number of unordered selections with repetitions allowed of five things from five possibilities, that is, \( \binom{5 + 5 - 1}{5} = \binom{9}{4} \). The same number counts the distributions of the copies of the second volume so the answer is \( \binom{9}{4}^2 = 15876 \).

(d) What is the number of ways to place the books so that no shelf is empty, each shelf only contains copies of the same volume and exactly two of the shelves contain copies of the first volume? (3 marks)

The number of ways to distribute the copies of the first volume between 2 given shelves (with each shelf containing at least one book) equals the number of solutions of the equation \( y_1 + y_2 = 10 \), where each \( y_i \) is a positive integer. The number of such solutions is \( \binom{8 + 2 - 1}{8} = 9 \). Similarly, the number of ways to distribute the copies of the second volume between 3 shelves equals the number of solutions of the equation \( y_1 + y_2 + y_3 = 10 \), which is \( \binom{7 + 3 - 1}{7} = \binom{9}{2} = 36 \).

Thus, taking into account the number of ways to choose two shelves for the first volume we find that the final answer is given by \( \binom{5}{2} \times 9 \times 36 = 3240 \).
2. For any values of the Boolean variables $x, y, z$ consider the proposition
\[ \exists a \in \{0, 1\} (xy = az). \]

The Boolean function $f(x, y, z)$ takes value 1 if the proposition is true and 0 otherwise.

(a) Complete the table of values for the function $f(x, y, z)$.  

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(b) Using the previous part or otherwise, show that $f(x, y, z)$ can be represented by a switching circuit with exactly three switches.

The table shows that the only zero value of the function is taken at $x = y = 1$ and $z = 0$. Therefore, the function can be written as \( f(x, y, z) = x' \lor y' \lor z \), because the Boolean expression \( x' \lor y' \lor z \) is zero if and only if $x = y = 1$ and $z = 0$. Hence, the function can be represented by a switching circuit with the switches $x'$, $y'$ and $z$ connected in parallel.

Alternatively, the Boolean expression $f(x, y, z) = x' \lor y' \lor z$ can be found by an application of the Karnaugh map method.
(c) Recall that the Fibonacci numbers $F_n$ are defined by the recurrence relation
\[ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2 \text{ with } F_0 = 0 \text{ and } F_1 = 1. \]
Use mathematical induction to prove that
\[ F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1} \]
for all $n \geq 1$. (3 marks)

Let $S(n)$ be the statement. Then $S(1)$ is true since $F_1^2 = 1$ and $F_1 F_2 = 1$. Suppose that $S(n)$ holds for some $n \geq 1$. Then by the induction hypothesis,
\[ F_1^2 + F_2^2 + \cdots + F_n^2 + F_{n+1}^2 = F_n F_{n+1} + F_{n+1}^2. \]
Using the recurrence relation for the Fibonacci numbers we find
\[ F_n F_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1}) = F_{n+1}F_{n+2}. \]
Hence, $S(n+1)$ holds. Thus, $S(n)$ is true for all $n \geq 1$. 
(d) The Lucas numbers $L_n$ are defined by the recurrence relation
$\quad L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with $L_0 = 2$ and $L_1 = 1$. Find a closed form for the generating function
$\quad L(z) = \sum_{n=0}^{\infty} L_n z^n$.

(3 marks)

After multiplying $L(z)$ by $z$ and $z^2$ we find that
$\quad L(z) = L_0 + L_1 z + L_2 z^2 + L_3 z^3 + \cdots$
$\quad zL(z) = \quad L_0 z + L_1 z^2 + L_2 z^3 + \cdots$
$\quad z^2L(z) = \quad L_0 z^2 + L_1 z^3 + \cdots$.

Then using the recurrence relation $L_n - L_{n-1} - L_{n-2} = 0$, we obtain
$\quad (1 - z - z^2)L(z) = L_0 + (L_1 - L_0)z$.

Taking into account the initial values $L_0 = 2$ and $L_1 = 1$, we then have
$\quad (1 - z - z^2)L(z) = 2 - z$.

Hence the closed form of the generating function is
$\quad L(z) = \frac{2 - z}{1 - z - z^2}$.
3. Consider the recurrence relation $a_n = -a_{n-1} + 8a_{n-2} + 12a_{n-3}$, $n \geq 3$.

(a) Write the corresponding characteristic equation and verify that one of its roots equals 3. (2 marks)

The characteristic equation is $\lambda^3 + \lambda^2 - 8\lambda - 12 = 0$. Taking $\lambda = 3$ we get $27 + 9 - 24 - 12 = 0$ so that 3 is a root of the equation.

(b) Find all roots of the characteristic equation. (2 marks)

Since 3 is a root of the characteristic polynomial, we can factorize

$$\lambda^3 + \lambda^2 - 8\lambda - 12 = (\lambda - 3)(\lambda^2 + a\lambda + b).$$

Comparing the constant terms and the coefficients of $\lambda^2$ we find $a = b = 4$. Therefore, the characteristic equation reads

$$(\lambda - 3)(\lambda + 2)^2 = 0.$$ 

The roots of the equation are $\lambda = -2$ occurring with multiplicity 2 and $\lambda = 3$ with multiplicity 1.
(c) Show that the particular solution of the recurrence relation subject to the initial conditions \( a_0 = a_1 = 1 \) and \( a_2 = 17 \) is found by \( a_n = n(-2)^n + 3^n \). (2 marks)

By the previous part, the general solution of the recurrence relation is

\[ a_n = (A n + B)(-2)^n + C 3^n, \]

where \( A, B, C \) are arbitrary constants. The initial conditions give \( B + C = 1 \), \((-2)(A + B) + 3C = 1\) and \(8A + 4B + 9C = 17\). Eliminating \( A \) from the last two relations gives \(-4B + 21C = 21\). Together with the first relation this gives \( B = 0 \) and \( C = 1 \). Hence \( A = 1 \) so that the particular solution is \( a_n = n(-2)^n + 3^n \).

Alternatively, we can verify directly that the sequence \( a_n = n(-2)^n + 3^n \) satisfies the recurrence relation and the initial conditions.

(d) Find a closed form of the generating function \( G(z) \) of the sequence \( a_n \) given in part (c). (2 marks)

We have

\[
G(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (n(-2)^n + 3^n) z^n = \sum_{n=1}^{\infty} n(-2z)^n + \sum_{n=0}^{\infty} (3z)^n \\
= (-2z) \sum_{n=0}^{\infty} (n + 1)(-2z)^n + \sum_{n=0}^{\infty} (3z)^n = -\frac{2z}{(1+2z)^2} + \frac{1}{1-3z}.
\]
(e) Show that all solutions of the recurrence relation $a_n = -a_{n-1} + 8a_{n-2} + 12a_{n-3}$, $n \geq 3$, satisfying the additional condition $a_2 = a_1 + 6a_0$ are also solutions of a homogeneous recurrence relation of degree two. \hfill (2 marks)

As we found in part (c), the general solution of the recurrence relation is

$$a_n = (A n + B)(-2)^n + C 3^n,$$

where $A, B, C$ are arbitrary constants. The additional condition implies

$$8A + 4B + 9C = -2A - 2B + 3C + 6B + 6C,$$

so that $A = 0$. Hence the sequence $a_n$ takes the form

$$a_n = B(-2)^n + C 3^n.$$

This sequence satisfies the homogeneous recurrence relation of degree two whose characteristic equation has roots $-2$ and $3$. That is, $a_n$ is a solution of the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ with $n \geq 2$. 

You may use next pages for your answers