1. Define $f : \mathbb{N} \rightarrow \mathbb{N}$ by $f(x) = x + 1$. Determine whether or not $f$ is
   (a) one-to-one; (b) onto.

   **Solution.**
   (a) If $x \neq y$, then clearly $x + 1 \neq y + 1$ so that $f(x) \neq f(y)$. Hence $f$ is one-to-one.
   (b) Since $x + 1 \geq 1$ for all $x \in \mathbb{N}$, we see that there is no $x \in \mathbb{N}$ such that $f(x) = x + 1 = 0$ and hence $f$ is not onto.

2. Each of the following sets of pairs may or may not represent a function from \{1, 2, 3\} to \{a, b, c, d\}.
   \[
   \{ (1, d), (2, b), (3, d), \}\quad \{ (1, c), (2, a), (3, b), \}\quad \{ (1, a), (3, b), \}
   \quad \{ (2, b), (3, c), (1, d) \}
   \]

   (i) Identify the sets which represent functions and determine which of these are one-to-one.
   (ii) Explain clearly why each of the sets does or does not represent a function.
   (iii) Explain clearly why each of the sets does or does not represent a one-to-one function.

   **Solution.**
   (a) The set \{ (1, d), (2, b), (3, d) \} represents a function since each of the elements 1,2,3 appears exactly once as the first term of an ordered pair. It is not one-to-one since d appears more than once as the second term of an ordered pair.
   (b) The set \{ (1, c), (2, a), (3, b) \} represents a function since each of the elements 1,2,3 appears exactly once as the first term of an ordered pair. It is one-to-one since none of the elements a,b,c,d appear more than once as second terms of ordered pairs.
   (c) The set \{ (1, a), (3, b) \} does not represent a function on the set 1,2,3 since 2 does not appear as the first term of an ordered pair.
   (d) The set \{ (1, a), (1, c), (3, d) \} does not represent a function since 1 appears more than once as the first term of an ordered pair. (Also, 2 does not appear as the first term of an ordered pair).
   (e) The set \{ (2, b), (3, c), (1, d) \} represents a function since each of 1,2,3 appears exactly once as the first term of an ordered pair. It is one-to-one since none of a,b,c,d appear more than once as the second term of an ordered pair.

3. (i) Let $A = \{-1, 2, 3, 5, 7, 11\}$ and let $B = \{1, 2, \ldots, 200\}$. Is the function $f : A \rightarrow B$ given by $f(x) = x^2$ one-to-one?

   (ii) Now suppose that $A = \{-2, -1, 2, 3, 5, 7, 11\}$ and $B = \{1, 2, \ldots, 200\}$. Is the function $f : A \rightarrow B$ given by $f(x) = x^2$ one-to-one?

   **Solution.**
   (i) It is easy to see that different elements in $A$ are mapped to different elements in $B$, so the given function $f$ is one-to-one.
   (ii) Since $f(-2) = f(2) = 4$, it follows that the given function $f$ is not one-to-one.
4. Use arrow diagrams to write down all the functions from the set \( \{1, 2\} \) to the set \( \{a, b, c\} \). How many are there? How many one-to-one functions and how many onto functions?

**Solution.**

The \( 3^2 = 9 \) functions are

![Diagram of functions from \( \{1, 2\} \) to \( \{a, b, c\} \)]

There are 6 one-to-one functions. Since there are more elements in the second set, there are no onto functions.

5. Let \( A = \{1, 2, 3\} \) and \( B = \{a, b, c\} \). Write down all the one-to-one correspondences between \( A \) and \( B \).

**Solution.**

The six one-to-one correspondences between \( A \) and \( B \) are

![Diagram of six one-to-one correspondences between \( A \) and \( B \)]
6. For each of the following functions, determine whether it is one-to-one and/or onto.

(i) \( f : \mathbb{Z} \to \mathbb{Z} \) given by \( f(x) = x^2 \).
(ii) \( f : \mathbb{N} \to \mathbb{Z} \) given by \( f(x) = x^2 \).
(iii) \( f : \mathbb{N} \to \mathbb{N} \) given by \( f(x) = x^2 \).
(iv) \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x^2 \).
(v) \( f : \mathbb{R}^+ \to \mathbb{R} \) given by \( f(x) = x^2 \).
(vi) \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) given by \( f(x) = x^2 \).

In this question \( \mathbb{R}^+ = \{ x \in \mathbb{R} \mid x > 0 \} \).

**Solution.**

(i) \( f \) is not one-to-one since \( f(1) = f(-1) = 1 \). \( f \) is not onto since \( f(x) = x^2 \) is always positive and so for \(-1 \in \mathbb{Z} \), there is no \( x \in \mathbb{Z} \) with \( f(x) = -1 \).

(ii) If \( f(x) = f(y) \) for \( x, y \in \mathbb{N} \), then \( x^2 = y^2 \) and so \( x = y \) since both \( x, y \in \mathbb{N} \). Hence \( f \) is one-to-one. \( f \) is not onto since \( f(x) = x^2 \) is always positive and so for \(-1 \in \mathbb{Z} \), there is no \( x \in \mathbb{N} \) with \( f(x) = -1 \).

(iii) Again \( f \) is one-to-one. But \( f \) is not onto, since for \( 2 \in \mathbb{N} \), there is no \( x \in \mathbb{N} \) such that \( f(x) = x^2 = 2 \).

(iv) \( f \) is not one-to-one since \( f(1) = f(-1) = 1 \). \( f \) is not onto since \( f(x) = x^2 \) is always positive and so for \(-1 \in \mathbb{R} \), there is no \( x \in \mathbb{R} \) with \( f(x) = -1 \).

(v) If \( f(x) = f(y) \) for \( x, y \in \mathbb{R}^+ \), then \( x^2 = y^2 \) and so \( x = y \) since both \( x, y \) are positive. Hence \( f \) is one-to-one. \( f \) is not onto since \( f(x) = x^2 \) is always positive and so for \(-1 \in \mathbb{R} \), there is no \( x \in \mathbb{R}^+ \) with \( f(x) = -1 \).

(vi) Again \( f \) is one-to-one. For any \( y \in \mathbb{R}^+ \), we see that \( x = \sqrt{y} \) is in \( \mathbb{R}^+ \) and that \( f(x) = (\sqrt{y})^2 = y \). Hence \( f \) is onto.

7. Let \( C_{n,k} \) be the set of subsets of \( \{1, ..., n\} \) of size \( k \).

(i) Describe explicitly \( C_{5,2} \). What is the cardinality of \( C_{5,2} \)?
(ii) Describe explicitly \( C_{4,2} \cup C_{4,1} \). What is the cardinality of \( C_{4,2} \cup C_{4,1} \)?
(iii) Define \( f : C_{5,2} \to C_{4,2} \cup C_{4,1} \) by \( f(A) = A \setminus \{5\} \). Describe \( f \) explicitly on the elements of \( C_{5,2} \).

**Solution.**

(i) We have

\[
C_{5,2} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}\}.
\]

Therefore the cardinality of \( C_{5,2} \) is ten.

(ii) We have

\[
C_{4,2} \cup C_{4,1} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1\}, \{2\}, \{3\}, \{4\}\}.
\]

Therefore the cardinality of \( C_{4,2} \cup C_{4,1} \) is also ten.

(iii) \( f \) acts in the following way: \( f(\{1,2\}) = \{1,2\}, f(\{1,3\}) = \{1,3\}, f(\{1,4\}) = \{1,4\}, f(\{1,5\}) = \{1\}, f(\{2,3\}) = \{2,3\}, f(\{2,4\}) = \{2,4\}, f(\{2,5\}) = \{2\}, f(\{3,4\}) = \{3,4\}, f(\{3,5\}) = \{3\}, f(\{4,5\}) = \{4\} \).
1. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Which of the following arrow diagrams determine functions from $A$ to $B$? Which of them determine a one-to-one functions?

**Solution.**

(i) The arrow diagram does not determine a function, since the element $a$ in $A$ is not assigned (or sent) to a unique element in $B$, (i.e., there are two arrows coming out of $a$, one pointing to 1 and the other to 3).

(ii) The arrow diagram determine a function from $A$ to $B$ and since distinct elements in $A$ are mapped to distinct elements in $B$, the function is one-to-one.

(iii) The arrow diagram determine a function from $A$ to $B$. Since both $b$ and $c$ are mapped to the same element 3, (i.e., the arrows coming out of $b$ and $c$ have the same end points 3), the function is not one-to-one.

(iv) The arrow diagram does not determine a function, since there is an element $c$ in $A$ which is not assigned (or sent) to any element in $B$, (i.e., there is no arrow coming out of $c$).

2. Use arrow diagrams to write down all the functions from the set $\{1, 2\}$ to $\{a, b\}$.

(i) Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Let $f : A \rightarrow B$ be the function given by the 4-tuple $(b, a, c, a)$. Draw the arrow diagram of $f$. Is $f$ injective? Is $f$ surjective?

(ii) Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Does the following set of pairs

$$F = \{(a, 2), (b, 1), (c, 2)\}$$

represent a function $g$ from $A$ to $B$.

If so, is $g$ injective? Is $g$ surjective?

**Solution.**

(i) The functions are:
(ii) The arrow diagram is as shown:

Since 2 and 4 are assigned to the same element \( a \) in \( B \), \( f \) is not injective. Since each element in \( B \) is assigned to some element in \( A \), \( f \) is surjective.

(iii) Since each element in \( A \) appears once in \( F \), \( F \) represents a function \( g \) from \( A \) to \( B \). Since 2 appears twice in \( F \), \( g \) is not injective. Since 3 does not appear in \( F \), \( g \) is not surjective.

OR: The arrow diagram is as shown:

Since \( a \) and \( c \) are mapped to the same element 2 in \( B \), \( g \) is not injective. Since there is no arrow ending at 3, \( g \) is not surjective.

3. Let \( A = \{0, 1, 2, 3\} \) and \( B = \{0, 1, 2, 3, 4, 5\} \), and consider the function \( f : A \to B \) with rule \( f(x) = x^2 - 4x + 4 \) for all \( x \in A \).

(i) Draw an arrow diagram to represent \( f \). Write \( f \) as a set of ordered pairs of integers.

(ii) Is \( f \) injective? Give reasons.

(iii) Find the image of \( f \). Is \( f \) surjective? Give reasons.

(iv) Find a set \( C \) such that with the same domain \( A \), \( f : A \to C \), \( f(x) = x^2 - 4x + 4 \) is surjective.

Solution.

(i) We see that \( f(0) = 4 \), \( f(1) = 1 \), \( f(2) = 0 \), \( f(3) = 1 \).

The arrow diagram is as shown:

\[ f \text{ as a set of ordered paired of integers: } \]

\[ f = \{(0, 4), (1, 1), (2, 0), (3, 1)\}. \]

(ii) The function is not injective since \( f(1) = f(3); \) that is, distinct elements of \( A \) do not map to distinct elements of \( B \).

(iii) The image of \( f \) is \( \{0, 1, 4\} \). The function is not surjective since there is no element of \( A \) which maps to 2 (or 3 or 5).
(iv) If $C = \{0, 1, 4\}$ then $f : A \rightarrow C$ with the rule $f(x) = x^2 - 4x + 4$ will be surjective, since $0 = f(2), 1 = f(1), 4 = f(0)$.

4. Use arrow diagrams to write down all the functions from the set $\{1, 2, 3, 4\}$ to the set $\{a, b\}$. How many are there? How many one-to-one functions and how many onto functions?

**Solution.**

The $2^4 = 16$ functions are:

![Diagram of functions](image)

There are 14 onto functions. Since there are more elements in the first set, there is no one-to-one function.