1. You have a deck of fifty-two cards.
   (i) How many ways are there of choosing a hand of five cards?
   (ii) How many of them contain the queen of hearts?
   (iii) In how many ways can four hands of five cards each be given to four players, if we care which hand goes to which player?
   (iv) In how many ways can four hands of five cards be selected from the deck? (This time we don’t care which hand goes to which player.)

   [Note that, as with all word problems, there is some chance of differing interpretations here. See the note on page ix of the text.]

   Solution.

   (i) The number of ways of choosing a hand of five cards is \( \binom{52}{5} = \frac{52!}{47!5!} \).

   (ii) If one of the five cards is the queen of hearts, then the problem is the same as choosing 4 cards from 51. The answer is thus

   \[
   \binom{51}{4} = \frac{51!}{47!4!} = 249900.
   \]

   (iii) There are \( \binom{52}{5} \) ways of giving the first hand of five cards to the first player. Then there are 47 cards left so that there are \( \binom{47}{5} \) ways of giving the second hand to the second player. Similarly, there are \( \binom{42}{5} \) ways of giving the third hand to the third player and \( \binom{37}{5} \) ways of giving the fourth hand to the fourth player. Hence the number of ways that four hands of five cards can be given to four players is

   \[
   \binom{52}{5} \binom{47}{5} \binom{42}{5} \binom{37}{5} = \frac{52!}{32!5!5!5!5!}.
   \]

   Alternatively, this can be written as a multinomial coefficient, where the left-over cards (32 of them) are considered as belonging to a fifth hand. The answer is then \( \binom{52}{5,5,5,5,32} \).

   (iv) The difference between this and the previous part is that now we don’t care which hand goes to which player. So what counts as 4! different ways of distributing hands in part (iii) only counts as one way here. Thus the answer is that for part (iii) divided by 4!, i.e.

   \[
   \frac{52!}{32!5!5!5!5!4!}.
   \]

2. Consider the set \{a, b, c, d, e, f\}. How many ways are there of choosing four letters from this set
   (i) if no letter is chosen twice?
   (ii) if repetitions are allowed?
Solution.

(i) If no letter is chosen twice, the number of ways of choosing four letters from the given set is \(\binom{6}{4} = 15\).

(ii) If repetitions are allowed, then the number of ways of choosing four letters from the given set is \(\binom{4 + 6 - 1}{4} = \binom{9}{4} = 126\).

3. (i) How many different outcomes are possible if seven identical dice are thrown? (An outcome is the collection of numbers, with repetition, visible on the top faces of the dice.)

(ii) Given a large supply of jelly beans of 10 different colours, how many ways are there to make up a bag of 5 jelly beans?

Solution.

(i) Since each die has six faces, we see that the number of possible outcomes if seven identical dice are thrown, is \(\binom{7 + 6 - 1}{7} = \binom{12}{7} = \frac{12!}{7!5!} = 792\).

(ii) The number of ways is \(\binom{5 + 10 - 1}{5} = \binom{14}{5} = \frac{14!}{5!9!} = 2002\).

4. How many distinguishable arrangements are there of the letters in the words

(i) imperseverant,   (ii) myristicivorous,

(iii) indistinguishable,   (iv) sociological?

Solution.

(i) Since there are 3 e’s, 2 r’s and 1 of each of the other 8 letters, the number of distinguishable arrangements of the letters in the word imperseverant is \(\frac{13!}{3!2!}\).

(ii) Since there are 2 r’s, 3 i’s, 2 s’s, 2 o’s and 1 of each of the other 6 letters, the number of distinguishable arrangements of the letters in the word myristicivorous is \(\frac{15!}{2!3!2!2!}\).

(iii) There are 4 i’s, 2 n’s, 2 s’s and 1 of each of the other 9 letters and so the number of distinguishable arrangements of the letters in the word indistinguishable is \(\frac{17!}{4!2!2!2!}\).

(iv) There are 1 s, 3 o’s, 2 c’s 2 i’s, 2 l’s, 1 g and 1 a so that the number of distinguishable arrangements of the letters in the word sociological is \(\frac{12!}{3!2!2!2!}\).

Note: All of these answers can be written as multinomial coefficients. For example, the answer to part (i) is \(\binom{13}{3, 2, 1, 1, 1, 1, 1, 1, 1, 1}\).

Note: “Imperseverant” does mean “not persevering”. It is in the big Oxford dictionary (available online through the Uni Library), but there is only one recorded mention of the word.

“Myristicivorous” is only doubtfully a word; very likely no-one has used it seriously; it is supposed to mean “nutmeg-eating”. It is not in the big Oxford dictionary. Search Google for myristicivorous Chambers for a comment on this word.

5. Suppose 20 people are divided into 6 different committees (labelled \(C_1\) to \(C_6\)). Suppose that committee \(C_1\) is to have 3 people, committee \(C_2\) is to have 4 people, committee \(C_3\) is to have 4 people, committee \(C_4\) is to have 2 people, committee \(C_5\) is to have 3 people and committee \(C_6\) is to have 4 people. How many arrangements are there?

Solution.

The total number of arrangements is \(\frac{20!}{3!4!4!2!3!4!} = \binom{20}{3, 4, 4, 2, 3, 4}\).
6. In how many ways can 15 distinct balls be placed in 4 boxes so that the first box contains 5 balls, the second box contains 3 balls, the third box contains 4 balls and the fourth box contains 3 balls?

Solution.

The total number of ways is

\[
\binom{15}{5} \binom{10}{3} \binom{7}{4} \binom{3}{3} = \frac{15!}{5! \cdot 3! \cdot 4! \cdot 3!} = \binom{15}{5,3,4,3}.
\]

7. Determine the coefficient of

(i) \(x_1^2x_2x_3\) in \((x_1 + x_2 + x_3)^4\)

(ii) \(x_1^2x_2^3x_3^2\) in \((x_1 + x_2 + x_3)^7\).

Solution.

(i) The coefficient of \(x_1^2x_2x_3\) in the expansion of \((x_1 + x_2 + x_3)^4\) is

\[
\binom{4}{2,1,1} = \frac{4!}{2! \cdot 1! \cdot 1!} = 12.
\]

(ii) The coefficient of \(x_1^2x_2^3x_3^2\) in the expansion of \((x_1 + x_2 + x_3)^7\) is

\[
\binom{7}{2,3,2} = \frac{7!}{2! \cdot 3! \cdot 2!} = 210.
\]
Problem Set 5

1. (i) How many words can be formed using all the letters of the word ANTIDISESTABLISHMENTARIANISM?
(ii) How many of the words in (i) have all the I’s together?
(iii) Find the coefficient of \(x_1^2x_2^4x_3^3x_4^2\) in the expansion of \((x_1+x_2+x_3+x_4)^{11}\).

Solution.

(i) There are 28 letters: 4 As, 3Ns, 3Ts, 5Is, 4Ss, 2Es, 2Ms and 1 of each of D, B, L, H and R. The number of words is therefore
\[
\frac{28!}{4!3!3!5!4!2!2!}.
\]

(ii) If the I’s are together, we can treat them as one big letter, and so the number of words is
\[
\frac{24!}{4!3!3!4!2!2!}.
\]

(iii) The coefficient is
\[
\frac{11!}{2!4!3!2!}.
\]

Note: The answers can be given as multinomial coefficients. For example, the answer for part (i) is \(\binom{28}{4,3,3,5,4,2,2,1,1,1,1,1}\).

Note: “Antidisestablishmentarianism” was for a long time considered to be the longest word actually used in English. It means a movement in England against those who wished to disestablish the “Established Church” (the Church of England). For other long words, see http://www.askoxford.com/asktheexperts/faq/aboutwords/longestword

2. Suppose that there are 8 different kinds of doughnuts in a coffee shop.
(i) How many ways can you buy 5 doughnuts of different kinds?
(ii) How many ways can you buy 5 doughnuts?

Solution.

(i) Since there are 8 different kinds of doughnuts, the number of ways you can buy 5 doughnuts of different kinds is \(\binom{8}{5} = 56\).

(ii) This is choosing 5 things from 8 types with repetition, and so the number of ways is \(\binom{5+8-1}{5} = \binom{12}{5} = 792\).
In how many ways can we choose 4 novels and 3 biographies if there are 8 novels and 6 biographies from which to choose?

(ii) How many ways are there to choose 12 coins from a large supply of 10 cents, 20 cents, 50 cents and 1 dollar coins?

(iii) Find the coefficient of $x_1^2x_2^3x_3x_4^4$ in the expansion of $(x_1+x_2+x_3+x_4)^{10}$

Solution.

(i) There are $\binom{8}{4}$ ways of choosing the 4 novels and $\binom{6}{3}$ ways of choosing the 3 biographies and so the number of ways of choosing 4 novels and 3 biographies is $\binom{8}{4} \times \binom{6}{3} = 1400$.

(ii) I am choosing 12 things from 4 types, with repetition. Hence the number of ways is $\binom{12+4-1}{12} = \binom{15}{12} = \frac{15!}{12!3!} = 455$.

(iii) The coefficient is $\binom{10}{2,3,1,4} = \frac{10!}{2!3!1!4!} = 12600$.

How many ways can the letters of the word HULLABALOO be arranged?

(ii) How many arrangements of the letters of HULLABALOO begin with U and end with L?

(iii) How many arrangements of the letters of HULLABALOO contain the two letters HU next to each other in order?

(iv) How many arrangements of the letters of HULLABALOO contain the two letters HU next to each other (in either order)?

Solution.

(i) Since there are 1 H, 1 U, 3 Ls, 2 As, 1 B and 2 Os, the number of ways is $\frac{10!}{1!1!3!2!1!2!} = 151200$.

(ii) Since the arrangements begin with U and end with L, there are 1 H, 2 Ls, 2 As, 1 B and 2 Os to be rearranged and so the number of such arrangements is $\frac{8!}{1!2!2!1!2!} = 5040$.

(iii) Since HU must be next to each other in order, we treat HU as 1 letter, and so the number of such arrangements is $\frac{9!}{1!3!2!1!2!} = 15120$.

(iv) Since HU must be next to each other, we treat HU or UH as 1 letter, and so the number of such arrangements is $2 \times \frac{9!}{1!3!2!1!2!} = 30240$.

How many ways of arranging six $a$’s and ten $b$’s with no consecutive $a$’s?

(ii) In how many ways can 8 identical pens be distributed among 4 students?

(iii) In how many ways can 8 identical pens be distributed among 4 students so that each student receives at least one pen?

(iv) In how many ways can 8 identical pens and 10 identical pencils be distributed among 4 students?
(v) In how many ways can 8 identical pens and 10 identical pencils be distributed among 4 students if each student receives at least 1 pen and 1 pencil?

(vi) Twelve students are awaiting enrolment in some courses (each student is to enrolled in just one course). If 3 of them are to be enrolled in Mathematics, 4 in Computer Science and 5 in Physics, in how many ways can the enrolment be made?

Solution.

(i) We must separate $a$ by $b$. Since there are 10 $b$’s, there are 11 positions one can choose for the six $a$’s. Hence the number of possible ways is

$$\binom{11}{6} = 462.$$  

(ii) This is equivalent to selecting among the students 8 times. So we are selecting 8 times from 4 things with repetition. The answer is

$$\binom{8 + 4 - 1}{8} = \binom{11}{8} = \binom{11}{3} = 165.$$  

(iii) Since each student receives at least one pen, let us begin by giving one pen to each student. We then distribute the 4 remaining pens arbitrarily. The answer is

$$\binom{4 + 4 - 1}{4} = \binom{7}{4} = 35.$$  

(iv) Let us distribute the pens first and the pencils second. As above, we see that the number of ways of distributing the pens is

$$\binom{8 + 4 - 1}{8} = \binom{11}{8}$$

and similarly the number of ways of distributing the pencils is

$$\binom{10 + 4 - 1}{10} = \binom{13}{10}.$$  

Hence the number of ways that 8 pens and 10 pencils can be distributed among 4 students is

$$\binom{11}{8} \times \binom{13}{10} = 165 \times 286 = 47190.$$  

(v) If each student receives at least 1 pen and 1 pencil, then the possible ways is

$$\binom{4 + 4 - 1}{4} \times \binom{6 + 4 - 1}{6} = \binom{7}{4} \times \binom{9}{6} = 35 \times 84 = 2940.$$  

(vi) There are $\binom{12}{3}$ ways of choosing 3 students for Mathematics. Then there are $\binom{9}{4}$ ways of choosing 4 from the remaining 9 students for Computer Science and finally there is $\binom{5}{5}$ ways to choose the remaining 5 for Physics. Hence the possible ways the enrolment can be made is

$$\binom{12}{3} \times \binom{9}{4} \times \binom{5}{5} = \frac{12!}{3!4!5!} = 27720.$$  

This is equal to the multinomial coefficient $\binom{12}{3, 4, 5}$. 

5