Use induction to prove the following propositions.

For each of the given propositions, we let $S(n)$ be the given proposition, which is to be proved true for all integers greater than or equal to some specified integer, $n_0$ say. Then we show that

(a) $S(n_0)$ is true.
(b) $(\forall n \geq n_0) \; (S(n) \Rightarrow S(n+1))$ is true. [To show (b), we suppose that $S(n)$ is true and, assuming that $n \geq n_0$, we prove that $S(n+1)$ is true.]

Then we conclude that $S(n)$ is true for all positive integers $n \geq n_0$.

1. Prove that $2^n \geq n + 12$, for all integers $n \geq 4$.

Solution.

(a) When $n = 4$, $2^4 = 16$ which is clearly $\geq 4 + 12$. Thus $S(4)$ is true.

(b) Suppose that $S(n)$ is true, i.e. $2^n \geq n + 12$. Then

$$2^{n+1} = 2 \cdot 2^n \geq 2 \cdot (n + 12) \quad \text{(induction hypothesis)}$$

$$= 2n + 24 = (n + 1) + 12 + n + 11$$

$$> (n + 1) + 12,$$

so that $S(n+1)$ is true.

Hence $S(n)$ is true for all integers $n \geq 4$.

2. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for all positive integers $n$.

Solution.

(a) $S(1)$ is the proposition $1 = 1^2$, which is clearly true.

(b) Suppose that $S(n)$ is true, i.e. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. Then

$$1 + 3 + 5 + \cdots + (2n - 1) + (2(n + 1) - 1)$$

$$= 1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1)$$

$$= [1 + 3 + 5 + \cdots + (2n - 1)] + (2n + 1)$$

$$= n^2 + (2n + 1) \quad \text{(induction hypothesis)}$$

$$= n^2 + 2n + 1$$

$$= (n + 1)^2,$$

so that $S(n+1)$ is true.

Hence $S(n)$ is true for all positive integers $n$.

3. Prove that the sum of the first $n$ positive even integers is $n^2 + n$.

Solution.

$S(n)$ is the proposition

$$2 + 4 + 6 + \cdots + 2n = n^2 + n.$$

(a) $S(1)$ is clearly true as $2 = 1^2 + 1$. 

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(b) Suppose that $S(n)$ is true, i.e.

$$2 + 4 + 6 + \cdots + 2n = n^2 + n.$$ 

Then we have

$$2 + 4 + 6 + \cdots + 2n + 2(n + 1) \\
= [2 + 4 + 6 + \cdots + 2n] + 2(n + 1) \\
= (n^2 + n) + 2(n + 1) \quad \text{(induction hypothesis)} \\
= (n^2 + 2n + 1) + (n + 1) \\
= (n + 1)^2 + (n + 1),$$

which shows that $S(n + 1)$ is true.

Hence $S(n)$ is true for all positive integers $n$.

4. Prove that $2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n + 1)}{2}$, for all positive integers $n$.

Solution.

(a) $S(1)$ is the proposition $2 = \frac{1(3 \cdot 1 + 1)}{2}$, which is clearly true.

(b) Suppose that $S(n)$ is true. That is, suppose that

$$2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n + 1)}{2}.$$ 

Then

$$2 + 5 + 8 + \cdots + (3n - 1) + (3n + 2) \\
= \frac{n(3n + 1)}{2} + (3n + 2) \quad \text{(induction hypothesis)} \\
= \frac{3n^2 + 7n + 4}{2} \\
= \frac{(n + 1)(3n + 4)}{2} \\
= \frac{(n + 1)(3(n + 1) + 1)}{2},$$

which shows that $S(n + 1)$ is true.

Hence $S(n)$ is true for all positive integers $n$.

5. Prove that 6 divides $n(n^2 + 5)$ for all positive integers $n$.

Solution.

(a) When $n = 1$, $n(n^2 + 5) = 1(1 + 5) = 6$, which is clearly divisible by 6. Hence $S(1)$ is true.

(b) Suppose that $S(n)$ is true. That is, suppose that $n(n^2 + 5) = 6\ell$, for some integer $\ell$. Then

$$(n + 1)((n + 1)^2 + 5) = (n + 1)(n^2 + 5 + 2n + 1) \\
= n(n^2 + 5) + 3n^2 + 3n + 6 \\
= 6\ell + 3n^2 + 3n + 6 \quad \text{(induction hypothesis)} \\
= 6\ell + 3n(n + 1) + 6.$$ 

For each positive integer $n$, either $n$ or $n + 1$ is even so that each term on the right-hand side (of the last equality) is divisible by 6. Thus

$$(n + 1)((n + 1)^2 + 5)$$

is divisible by 6 and so $S(n + 1)$ is true.

Hence $S(n)$ is true for all positive integers $n$. 2
6. Prove that $11^n - 4^n$ is divisible by 7 for all positive integers $n$.

**Solution.**

(a) When $n = 1$, $11^n - 4^n = 7$, which is clearly divisible by 7. Hence $S(1)$ is true.

(b) Suppose that $S(n)$ is true. That is, suppose that $11^n - 4^n = 7\ell$, for some integer $\ell$. Then

\[
11^{(n+1)} - 4^{(n+1)} = 11^{(n+1)} - 11 \cdot 4^n + 11 \cdot 4^n - 4^{(n+1)}
= 11(11^n - 4^n) + 4^n(11 - 4)
= 11 \cdot 7\ell + 7 \cdot 4^n 
= 7(11\ell + 4^n),
\]

which shows that $11^{(n+1)} - 4^{(n+1)}$ is divisible by 7, and so $S(n+1)$ is true. Hence $S(n)$ is true for all positive integers $n$.

7. Prove that $5^n - 4n - 1$ is divisible by 16 for all positive integers $n$.

**Solution.**

(a) When $n = 1$, $5^n - 4n - 1 = 0$, which is clearly divisible by 16. Hence $S(1)$ is true.

(b) Suppose that $S(n)$ is true; that is, suppose that $5^n - 4 - 1 = 16\ell$, for some integer $\ell$. Then

\[
5^{(n+1)} - 4(n + 1) - 1 = 5(5^n - 4n - 1) + 5(4n) + 5 - 4n - 4 - 1
= 5(16\ell) + 4(4n) 
= 16(5\ell + n),
\]

which shows that $5^{(n+1)} - 4(n + 1) - 1$ is divisible by 16, and so $S(n+1)$ is true. Hence $S(n)$ is true for all positive integers $n$.

8. Prove that for any integer $n \geq 1$, $\frac{(2n)!}{2^n}$ is an integer.

**Solution.**

(a) For $n = 1$, we see that $\frac{2!}{2} = 1$ and so $S(1)$ is true.

(b) Suppose that $S(n)$ is true; that is, suppose that

\[
\frac{(2n)!}{2^n} = \ell
\]

for some integer $\ell > 0$. Then

\[
\frac{[2(n + 1)]!}{2^{n+1}} = \frac{2(n + 1)(2n + 1)(2n)!}{2^{n+1}}
= (n + 1)(2n + 1) \cdot \frac{(2n)!}{2^n}
= (n + 1)(2n + 1)\ell 
= (n + 1)(2n + 1)\ell 
\]

and so $S(n+1)$ is true. Hence $S(n)$ is true for all positive integers $n$. 

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Problem Set 10

1. Use Mathematical Induction to show that:

   (i) The sum of the cubes of any three consecutive positive integers is divisible by 9.

   (ii) For all positive integers $n$,

   $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

Solution.

   (i) The sum of the cubes of any three consecutive positive integers is given by $n^3 + (n+1)^3 + (n+2)^3$.

   Let $S(n)$ be the proposition

   $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

   (a) For $n = 1$, $n^3 + (n+1)^3 + (n+2)^3 = 1 + 8 + 27 = 36$, which is clearly divisible by 9.

   (b) Suppose that $S(n)$ is true. That is, suppose that $n^3 + (n+1)^3 + (n+2)^3 = 9\ell$, for some integer $\ell$. Then

   $$(n + 1)^3 + (n + 2)^3 + (n + 3)^3 = (n + 1)^3 + (n + 2)^3 + n^3 + 3n^2 \cdot 3 + 3n \cdot 3^2 + 3^3$$

   $$= 9\ell + 9n^2 + 27n + 27 \quad (\text{induction hypothesis})$$

   $$= 9(\ell + n^2 + 3n + 3),$$

   which shows that $(n + 1)^3 + (n + 2)^3 + (n + 3)^3$ is divisible by 9, and so $S(n+1)$ is true.

   Hence $S(n)$ is true for all positive integers $n$.

   (ii) Let $S(n)$ be the given proposition.

   (a) Since

   $\frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1 + 1}$,

   $S(1)$ is true.

   (b) Suppose that $S(n)$ is true. That is, suppose that

   $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$. 4
Then we have
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+1+1)}
\]
\[
= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \quad \text{(induction hypothesis)}
\]
\[
= \frac{n(n+2) + 1}{(n+1)(n+2)}
\]
\[
= \frac{n^2 + 2n + 1}{(n+1)(n+2)}
\]
\[
= \frac{(n+1)^2}{(n+1)(n+2)}
\]
\[
= \frac{n+1}{(n+1)+1}.
\]
which shows that \(S(n+1)\) is true.
Hence \(S(n)\) is true for all positive integers \(n\).

2. Use Mathematical Induction to show that:

(i) \(n! > n^2\) for all integers \(n \geq 4\).

(ii) \(2^{2^n+1} + 1\) is divisible by 3 for all positive integers \(n\).

Solution.

(i) Let \(S(n)\) be the given proposition.

(a) For \(n = 4\), \(4! = 24\) which is clearly greater than \(4^2 = 16\). Thus \(S(4)\) is true.

(b) Suppose that \(S(n)\) is true for some \(n \geq 4\). That is, suppose that for some \(n \geq 4\),

\[n! > n^2\]

Then

\[(n+1)! = (n+1)n! > (n+1)n^2 \quad \text{(induction hypothesis)}
\]
\[
> (n+1)(n+1) \quad \text{(since } n \geq 4 \Rightarrow n^2 \geq 4n \geq n+1) 
\]
\[
= (n+1)^2,
\]
which shows that \(S(n+1)\) is true.
Hence \(S(n)\) is true for all integers \(n \geq 4\).
(ii) Let $S(n)$ be the given proposition.

(a) When $n = 1$, we see that $2^{2n+1} + 1 = 9$ which is clearly divisible by 3. Thus $S(1)$ is true.

(b) Suppose that $S(n)$ is true. That is, suppose that $2^{2n+1} + 1 = 3\ell$, for some integer $\ell$. Then

$$2^{2(n+1)+1} + 1 = 2^{2n+3} + 1$$
$$= 2^{2n+1}2^2 + 1$$
$$= 4(2^{2n+1} + 1) - 4 + 1$$
$$= 4 \times 3\ell - 3 \quad \text{ (induction hypothesis)}$$
$$= 3(4\ell - 1),$$

and so $2^{2(n+1)+1} + 1$ is divisible by 3. Thus $S(n+1)$ is true.

Hence $S(n)$ is true for all integers $n$.

3. Use Mathematical Induction to show that:

(i) For all positive integers $n$, $n! \geq 2^{n-1}$.

(ii) For all positive integers $n$, 

$$\frac{(n+1)(n+2)\cdots(2n)}{2^n}$$

is an integer.

Solution.

(i) Let $S(n)$ be the given proposition.

(a) When $n = 1$, we see that $1! = 2^{1-1}$ is clearly true. Hence $S(1)$ is true.

(b) Suppose that $S(n)$ is true, i.e. $n! \geq 2^{n-1}$. Then

$$(n+1)! = (n+1)n!$$
$$\geq (n+1)2^{n-1} \quad \text{ (induction hypothesis)}$$
$$\geq 2 \cdot 2^{n-1} \quad \text{ since } n \geq 1$$
$$\geq 2^n,$$

and so $S(n+1)$ is true.

Hence $S(n)$ is true for all integers $n$.

(ii) Let $S(n)$ be the given proposition.

(a) For $n = 1$, the expression becomes $\frac{2}{2}$ which is the integer 1. Thus $S(1)$ is true.

(b) Suppose that $S(n)$ is true; that is, suppose that

$$\frac{(n+1)(n+2)\cdots(2n)}{2^n} = \ell,$$

where $\ell$ is an integer. Then

$$\frac{(n+2)\cdots(2n)(2n+1)(2n+2)}{2^{n+1}}$$
$$= \frac{(n+1)(n+2)\cdots(2n)}{2^n} \times \frac{(2n+1)2(n+1)}{2(n+1)} = \ell(2n+1),$$

using the induction hypothesis. Since $\ell(2n+1)$ is an integer, it follows that $S(n+1)$ is true.

Hence $S(n)$ is true for all integers $n$. 

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4. Use Mathematical Induction to show that:

(i) For any integers \( n \geq 1 \), \( 13^n - 5^n \) is divisible by 8.

(ii) For any integer \( n \geq 1 \),

\[
\frac{5}{1 \cdot 2 \cdot 3} + \frac{6}{2 \cdot 3 \cdot 4} + \cdots + \frac{n + 4}{n(n + 1)(n + 2)} = \frac{n(3n + 7)}{2(n + 1)(n + 2)}.
\]

Solution.

(i) Let \( S(n) \) be the given proposition.

(a) For \( n = 1 \), we see that \( 13^1 - 5^1 = 8 \) which is clearly divisible by 8. Thus \( S(1) \) is true.

(b) Suppose that \( S(n) \) is true; i.e., suppose that for some \( n \geq 1 \),

\[
13^n - 5^n = 8\ell,
\]

for some \( \ell > 0 \). To show that \( S(n + 1) \) is true, we proceed as follows:

\[
13^{n+1} - 5^{n+1} = 13(13^n - 5^n) + 13 \cdot 5^n - 5^{n+1} = 13 \cdot 8\ell + 5^n(13 - 5) \quad \text{(induction hypothesis)}
\]

and so \( 13^{n+1} - 5^{n+1} \) is divisible by 8. Thus \( S(n + 1) \) is true whenever \( S(n) \) is true.

Hence \( S(n) \) is true for all integers \( n \geq 1 \).

(ii) Let \( S(n) \) be the given proposition.

(a) For \( n = 1 \),

\[
\text{LHS} = \frac{5}{1 \cdot 2 \cdot 3} = \frac{5}{6} \quad \text{and} \quad \text{RHS} = \frac{1 \cdot 10}{2 \cdot 2 \cdot 3} = \frac{5}{6}
\]

and so \( S(1) \) is true.

(b) Suppose that \( S(n) \) is true; that is, suppose that

\[
\frac{5}{1 \cdot 2 \cdot 3} + \frac{6}{2 \cdot 3 \cdot 4} + \cdots + \frac{n + 4}{n(n + 1)(n + 2)} = \frac{n(3n + 7)}{2(n + 1)(n + 2)}.
\]

Then

\[
\frac{5}{1 \cdot 2 \cdot 3} + \frac{6}{2 \cdot 3 \cdot 4} + \cdots + \frac{n + 4}{n(n + 1)(n + 2)} + \frac{n + 5}{(n + 1)(n + 2)(n + 3)} = \frac{n(3n + 7)}{2(n + 1)(n + 2)} + \frac{n + 5}{(n + 1)(n + 2)(n + 3)}
\]

\[
= \frac{n(3n + 7)(n + 3) + 2(n + 5)}{2(n + 1)(n + 2)(n + 3)}
\]

\[
= \frac{3n^3 + 16n^2 + 23n + 10}{2(n + 1)(n + 2)(n + 3)}
\]

\[
= \frac{(n + 1)(3n^2 + 13n + 10)}{2(n + 1)(n + 2)(n + 3)}
\]

\[
= \frac{(n + 1)(n + 1)(3n + 10)}{2(n + 2)(n + 3)}
\]

and so \( S(n + 1) \) is true.

Hence \( S(n) \) is true for all integers \( n \).