Tutorial 12

1. (i) Show that \( x_n = 6 \cdot 2^n - 4 \) is a solution to the recurrence relation
\[ x_n = 3x_{n-1} - 2x_{n-2}. \]

(ii) Show that \( x_n = (3^{n+1} - 1)/2 \) is a solution to the recurrence relation
\[ x_{n+1} - x_n = 3^{n+1}. \]

(iii) Show that \( x_n = n! \) is a solution to the recurrence relation
\[ x_n - n(n-1)x_{n-2} = 0. \]

Solution.

(i) If \( x_n = 6 \cdot 2^n - 4 \) then \( x_{n-1} = 6 \cdot 2^{n-1} - 4 \) and \( x_{n-2} = 6 \cdot 2^{n-2} - 4 \) so that
\[ 3x_{n-1} - 2x_{n-2} = 3(6 \cdot 2^{n-1} - 4) - 2(6 \cdot 2^{n-2} - 4) = 3 \cdot 6 \cdot 2^{n-1} - 12 - 2 \cdot 6 \cdot 2^{n-2} + 8 = 9 \cdot 2^n - 3 \cdot 2^n - 4 = 6 \cdot 2^n = 4 = x_n. \]
Hence \( x_n = 6 \cdot 2^n - 4 \) is a solution of the recurrence relation.

(ii) If \( x_n = (3^{n+1} - 1)/2 \), then \( x_{n+1} = (3^{n+2} - 1)/2 \) so that
\[ x_{n+1} - x_n = \frac{3^{n+2} - 1}{2} - \frac{3^{n+1} - 1}{2} = \frac{3^{n+2} - 3^{n+1}}{2} = 3^{n+1}. \]
Hence \( x_n = (3^{n+1} - 1)/2 \) is a solution of the recurrence relation.

(iii) If \( x_n = n! \) then \( x_{n-2} = (n-2)! \) and so
\[ x_n - n(n-1)x_{n-2} = n! - n(n-1)(n-2)! = n! - n! = 0. \]
Hence \( x_n = n! \) is a solution to the recurrence relation.

2. Solve the following recurrence relations:

(i) \( x_n - 5x_{n-1} + 6x_{n-2} = 0, \ n \geq 2, \ x_0 = 3, \ x_1 = 7 \)

(ii) \( x_n - 8x_{n-1} + 16x_{n-2} = 0, \ n \geq 2, \ x_0 = 3, \ x_1 = 20 \).

Solution.

(i) The characteristic equation is \( \lambda^2 - 5\lambda + 6 = 0 \). That is, \( (\lambda - 2)(\lambda - 3) = 0 \) and so \( \lambda = 2 \) or \( \lambda = 3 \). Thus the general solution is
\[ x_n = A2^n + B3^n, \]
for some constants \( A \) and \( B \). Using the initial conditions \( x_0 = 3 \) and \( x_1 = 7 \), we obtain \( 3 = A + B \) and \( 7 = 2A + 3B \). Solving yields \( A = 2 \) and \( B = 1 \). Hence the solution is
\[ x_n = 2 \cdot 2^n + 3^n. \]

(ii) The characteristic equation is \( \lambda^2 - 8\lambda + 16 = 0 \). That is, \( (\lambda - 4)^2 = 0 \) and so \( \lambda = 4 \) is a repeated root. Thus the general solution is
\[ x_n = A4^n + Bn4^n, \]
for some constants \( A \) and \( B \). Using the initial conditions \( x_0 = 3 \) and \( x_1 = 20 \), we obtain \( 3 = A \) and \( 20 = 4A + 4B \). Then \( A = 3 \) and \( B = 2 \). Hence the solution is
\[ x_n = 3 \cdot 4^n + 2n4^n = (3 + 2n)4^n. \]
3. Solve the following recurrence relations:

(i) \( x_n = 10x_{n-1} - 25x_{n-2} \), for \( n \geq 2 \), where \( x_0 = -1 \) and \( x_1 = 5 \).

(ii) \( x_{n+3} - 6x_{n+2} + 11x_{n+1} - 6x_n = 0 \), for \( n \geq 0 \), where \( x_0 = 1 \), \( x_1 = 0 \) and \( x_2 = -1 \).

Solution.

(i) The recurrence relation can be written as

\[
x_n - 10x_{n-1} + 25x_{n-2} = 0.
\]

Then the characteristic equation is

\[
\lambda^2 - 10\lambda + 25 = 0.
\]

That is, \((\lambda - 5)^2 = 0\) so that \(\lambda = 5\) is a repeated root with multiplicity 2. Thus the general solution to the recurrence relation is

\[
x_n = (A + Bn)5^n,
\]

where \(A\) and \(B\) are some constants. Now, using the initial values \(x_0 = -1\) and \(x_1 = 5\), we have

\[
-A = -1 \quad \text{and} \quad 5 = (A + B)5.
\]

Solving yields \(A = -1\) and \(B = 2\). Hence the solution is

\[
x_n = (-1 + 2n)5^n.
\]

(ii) The characteristic equation is

\[
\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.
\]

Then \((\lambda - 1)(\lambda - 2)(\lambda - 3) = 0\), and so \(\lambda = 1\), \(\lambda = 2\) or \(\lambda = 3\). Thus the general solution to the recurrence relation is

\[
x_n = A + B2^n + C3^n,
\]

for some constants where \(A\), \(B\) and \(C\). Now, using the initial values \(x_0 = 1\), \(x_1 = 0\), \(x_2 = -1\), we obtain

\[
\begin{align*}
A + B + C &= 1 \\
A + 2B + 3C &= 0 \\
A + 4B + 9C &= -1.
\end{align*}
\]

Solving this system of linear equations yields \(A = \frac{5}{2}\), \(B = -2\), and \(C = \frac{1}{2}\). Hence the solution is

\[
x_n = \frac{5}{2} - 2 \cdot 2^n + \frac{1}{2} \cdot 3^n.
\]

4. Solve the following recurrence relations:

(i) \( x_n = 4x_{n-1} - 3x_{n-2} \), where \( x_0 = 1 \) and \( x_1 = 2 \).

(ii) \( x_n = 3x_{n-1} - 3x_{n-2} + x_{n-3} \), where \( x_0 = 0 \), \( x_1 = 1 \) and \( x_2 = 3 \).

Solution.

(i) The recurrence relation can be written as

\[
x_n - 4x_{n-1} + 3x_{n-2} = 0.
\]

Then the characteristic equation is

\[
\lambda^2 - 4\lambda + 3 = 0.
\]

That is, \((\lambda - 1)(\lambda - 3) = 0\) so that \(\lambda = 1\) or \(\lambda = 3\). Thus the general solution for the given recurrence relation is

\[
x_n = A1^n + B3^n.
\]
for some constants $A$ and $B$. Now, using the initial values $x_0 = 1$ and $x_1 = 2$, we obtain
$1 = A + B$ and $2 = A + 3B$ which implies that $A = B = \frac{1}{2}$. Hence the solution is

$$x_n = \frac{1}{2} + \frac{1}{2} \cdot 3^n.$$  

(ii) The recurrence relation can be written as

$$x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3} = 0.$$  

Then the characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0.$$  

That is, $(\lambda - 1)^3 = 0$ so that $\lambda = 1$ is a repeated root with multiplicity 3. Thus the general solution for the recurrence relation is

$$x_n = A1^n + Bn1^n + Cn^21^n = A + Bn + Cn^2,$$

where $A$, $B$ and $C$ are some constants. Now, using the initial values $x_0 = 0$, $x_1 = 1$ and $x_2 = 3$, we have

$$0 = A$$
$$1 = A + B + C$$
$$3 = A + 2B + 4C.$$  

Solving the system yields $A = 0$, $B = C = \frac{1}{2}$. Hence the solution is

$$x_n = \frac{1}{2}(n + n^2) = \frac{1}{2}n(n + 1).$$
Problem Set 12

1. Solve the following recurrence relations:
   (i) \( x_n + 6x_{n-1} - 7x_{n-2} = 0 \) for all \( n \geq 2 \), with \( x_0 = 1 \) and \( x_1 = 2 \).
   (ii) \( x_{n+3} - 4x_{n+2} + 5x_{n+1} - 2x_n = 0 \) for all \( n \geq 0 \), with \( x_0 = 4 \), \( x_1 = 7 \) and \( x_2 = 17 \).

Solution.
   (i) The characteristic equation is \( \lambda^2 + 6\lambda - 7 = 0 \).
       That is, \( (\lambda - 1)(\lambda + 7) = 0 \) and so \( \lambda = 1 \) or \( \lambda = -7 \). Thus the general solution is
       \( x_n = A1^n + B(-7)^n \),
       for some constants \( A \) and \( B \). Using the initial conditions \( x_0 = 1 \) and \( x_1 = 2 \), we obtain
       \( 1 = A + B \) and \( 2 = A - 7B \). Then \( A = \frac{9}{5} \) and \( B = -\frac{4}{5} \). Hence the solution is
       \( x_n = \frac{9}{5} - \frac{4}{5}(-7)^n \).

   (ii) The characteristic equation is \( \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0 \).
       That is, \( (\lambda - 2)(\lambda - 1)^2 = 0 \) and so \( \lambda = 2 \) or \( \lambda = 1 \), a double root. Thus the general solution is
       \( x_n = A2^n + (B + Cn)1^n \),
       for some constants \( A \), \( B \) and \( C \). Using the initial conditions \( x_0 = 4 \), \( x_1 = 7 \) and \( x_2 = 17 \), we obtain
       \( 4 = A + B \),
       \( 7 = 2A + B + C \),
       \( 17 = 4A + B + 2C \).
       Solving this system of linear equations yields \( A = 7 \), \( B = -3 \) and \( C = -4 \). Hence the solution is
       \( x_n = 7 \cdot 2^n + (-3 - 4n)1^n = 7 \cdot 2^n - 3 - 4n \).

2. Solve the following recurrence relations:
   (i) \( x_{n+2} - 4x_{n+1} + 4x_n = 0 \) for all \( n \geq 0 \), with \( x_0 = 1 \) and \( x_1 = 4 \).
   (ii) \( x_{n+2} + 2x_{n+1} - 3x_n = 0 \) for all \( n \geq 0 \), with \( x_0 = 1 \) and \( x_1 = -1 \).

Solution.
   (i) The characteristic equation is \( \lambda^2 - 4\lambda + 4 = 0 \). That is, \( (\lambda - 2)^2 = 0 \) and so \( \lambda = 2 \) is a repeated root (root of multiplicity 2). Thus the general solution is
       \( x_n = (A + Bn)2^n \),
       for some constants \( A \) and \( B \). Using the initial conditions \( x_0 = 1 \) and \( x_1 = 4 \), we obtain
       \( 1 = A \) and \( 4 = 2(A + B) \). Then \( A = 1 \) and \( B = 1 \). Hence the solution is
       \( x_n = (1 + n)2^n \).

   (ii) The characteristic equation is \( \lambda^2 + 2\lambda - 3 = 0 \). That is, \( (\lambda - 1)(\lambda + 3) = 0 \) and so \( \lambda = 1 \) or \( \lambda = -3 \). Thus the general solution is
       \( x_n = A1^n + B(-3)^n \),
       for some constants \( A \) and \( B \). Using the initial conditions \( x_0 = 1 \) and \( x_1 = -1 \), we obtain
       \( 1 = A + B \) and \( -1 = A - 3B \). Solving yields \( A = \frac{1}{2} \) and \( B = \frac{1}{2} \). Hence the solution is
       \( x_n = \frac{1}{2} + \frac{1}{2}(-3)^n \).
3. Solve the following recurrence relations:

(i) \( x_n = -x_{n-1} + 6x_{n-2} \), where \( x_0 = 7 \) and \( x_1 = 4 \).

(ii) \( x_n = 6x_{n-1} - 9x_{n-2} \), where \( x_0 = -2 \) and \( x_1 = 6 \).

Solution.

(i) The recurrence relation can be written as

\[ x_n + x_{n-1} - 6x_{n-2} = 0. \]

Then the characteristic equation is

\[ \lambda^2 + \lambda - 6 = 0. \]

That is, \((\lambda + 3)(\lambda - 2) = 0\) so that \(\lambda = -3\) or \(\lambda = 2\). Thus the general solution to the recurrence relation is

\[ x_n = A(-3)^n + B2^n, \]

for some constants \(A\) and \(B\). Now, using the initial values \(x_0 = 7\) and \(x_1 = 4\), we have \(7 = A + B\) and \(4 = -3A + 2B\). Solving yields \(A = 2\) and \(B = 5\). Hence the solution is

\[ x_n = 2(-3)^n + 5 \cdot 2^n. \]

(ii) The recurrence relation can be written as

\[ x_n - 6x_{n-1} + 9x_{n-2} = 0. \]

The characteristic equation is

\[ \lambda^2 - 6\lambda + 9 = 0. \]

That is, \((\lambda - 3)^2 = 0\) so that \(\lambda = 3\) is a repeated root with multiplicity 2. Thus the general solution to the recurrence relation is

\[ x_n = (A + Bn)3^n, \]

where \(A\) and \(B\) are some constants. Now, using the initial values \(x_0 = -2\) and \(x_1 = 6\), we obtain \(-2 = A\) and \(6 = 3(A + B)\). Solving yields \(A = -2\) and \(B = 4\) and the solution is

\[ x_n = -2 \cdot 3^n + 4n \cdot 3^n = (-2 + 4n)3^n. \]

4. Solve the following recurrence relations:

(i) \( x_n = 5x_{n-1} - 6x_{n-2} \), where \( x_0 = 1 \) and \( x_1 = 1 \).

(ii) \( x_n - 5x_{n-1} + 8x_{n-2} - 4x_{n-3} = 0 \), where \( x_0 = 0 \), \( x_1 = 2 \) and \( x_2 = 4 \).

Solution.

(i) The recurrence relation can be written as

\[ x_n - 5x_{n-1} + 6x_{n-2} = 0. \]

Then the characteristic equation is

\[ \lambda^2 - 5\lambda + 6 = 0. \]

That is, \((\lambda - 2)(\lambda - 3) = 0\) so that \(\lambda = 2\) or \(\lambda = 3\). Thus the general solution to the recurrence relation is

\[ x_n = A2^n + B3^n, \]

for some constants \(A\) and \(B\). Now, using the initial values \(x_0 = 1\) and \(x_1 = 1\), we have \(1 = A + B\) and \(1 = 2A + 3B\). A simple calculation gives \(A = 2\) and \(B = -1\). Hence the solution is

\[ x_n = 2 \cdot 2^n - 3^n. \]
(ii) The characteristic equation of the recurrence relation is
\[ \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0. \]

Then \((\lambda - 1)(\lambda - 2)^2 = 0\) so that \(\lambda = 1\) is a root and \(\lambda = 2\) is a repeated root with multiplicity 2. Thus the general solution for the recurrence relation is
\[ x_n = A1^n + B2^n + Cn2^n, \]
for some constants \(A\), \(B\) and \(C\). Now, using the initial values \(x_0 = 0\), \(x_1 = 2\) and \(x_2 = 4\), we obtain
\[
\begin{align*}
0 &= A + B \\
2 &= A + 2B + 2C \\
4 &= A + 4B + 8C.
\end{align*}
\]
Solving this system of linear equations yields \(A = -4\), \(B = 4\) and \(C = -1\). Hence the solution is
\[ x_n = -4 + 4 \cdot 2^n - n \cdot 2^n. \]