# MATH1011 - APPLICATIONS OF CALCULUS

Improper Integrals - Week 12 Lect 2

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**More Improper Integrals**

*Example* 1. Investigate $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$.

Notice that the integrand is defined everywhere; there are no vertical asymptotes.
We split the integral at 0 and calculate

$$\lim_{a\to-\infty}\int_a^0 xe^{-x^2}\,dx$$

.

*Example* 1 (continued). Use substitution. Let $u=-x^2$. Then $\frac{du}{dx}=-2x$, so $du=-2x\,dx$ or $x\,dx=-\frac{1}{2}\,du$.
When $x=a$, $u=-a^2$ and when $x=0$, $u=0$.

$$\begin{aligned}\int_a^0 xe^{-x^2}\,dx &= \int_a^0 e^{-x^2}x\,dx\\ &= -\frac{1}{2}\int_{-a^2}^0 e^u\,du\\ &= -\frac{1}{2}\left[e^u\right]_{-a^2}^0\\ &= -\tfrac{1}{2}+\tfrac{1}{2}e^{-a^2}\to-\tfrac{1}{2} \quad \text{as} \quad a\to-\infty.\end{aligned}$$

*Example* 1 (continued). So

$$\int_{-\infty}^0 xe^{-x^2}\,dx=-\tfrac{1}{2}.$$

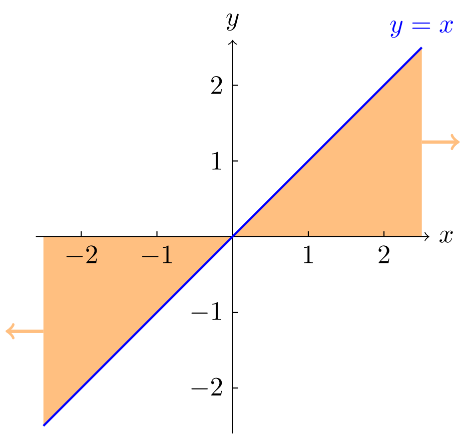
Since

$$\int_0^{\infty} xe^{-x^2}\,dx=-\int_{-\infty}^0 xe^{-x^2}\,dx=\frac{1}{2},$$

we have

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx=\int_{-\infty}^0 xe^{-x^2}\,dx+\int_0^{\infty} xe^{-x^2}\,dx=\tfrac{1}{2}-\tfrac{1}{2}=0.$$

Why couldn't we just say that, since the integrand $xe^{-x^2}$ was odd, $\int_{-\infty}^{\infty} xe^{-x^2}\,dx=0$?
We can't because arithmetic such as $-\infty+\infty=0$ is NOT valid.
For example, $\int_{-\infty}^{\infty} x\,dx$ does not exist.



**Application: Probability Densities**

Probability density functions provide a very good example of improper integrals.

A random variable takes values in the interval $[5, \infty)$.

Is there a value of $k$ such that $\frac{k}{(x-2)^2}$ is a probability density function on this interval? If so, find such $k$.

I.e. is there $k$ such that $\lim_{b\to\infty} \int_5^b \frac{k}{(x-2)^2} = 1$?

The function $\frac{1}{(x-2)^2}$ is defined throughout $[5, \infty)$.

Use substitution with $u = x - 2$, so $du = dx$.

When $x = 5$, $u = 3$ and when $x = b$, $u = b - 2$.

*Example* 1 (continued).

$$\begin{aligned}\int_5^b \frac{1}{(x-2)^2}\,dx &= \int_3^{b-2} u^{-2}\,du \\ &= \left[-u^{-1}\right]_3^{b-2} \\ &= -\frac{1}{b-2} + \frac{1}{3} \\ &\to \frac{1}{3} \text{ as } b \to \infty.\end{aligned}$$

Hence

$$\int_5^\infty \frac{k}{(x-2)^2}\,dx = \frac{k}{3} = 1 \text{ when } k = 3.$$

So $\frac{3}{(x-2)^2}$ is a probability density function on the interval $[5, \infty)$.

**Example**

**Question 2.** A random variable $x$ on the interval $[5, \infty)$ has a probability density function $\frac{3}{(x-2)^2}$.

Compute the probability that $x$ lies between 8 and 11.