

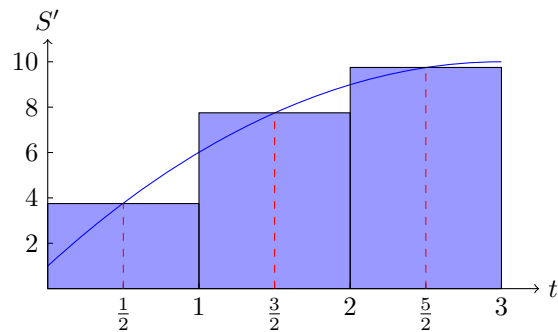
MATH1011 - APPLICATIONS OF CALCULUS

Finite sums and the definite integral - Week 9 Lect 2

7 May, 2010

The Definite Integral

Assume we take a three hour trip with the speed following the curve:



Let $S(t)$ be the distance travelled after t hours, in km. Then the speed is $S'(t)$.

Record the speed at some time in each hour:

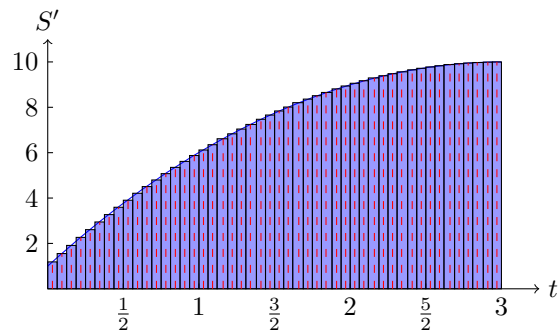
- t_1 in the first hour;
- t_2 in the second hour;
- t_3 in the third hour.

Recall: $Distance = Speed \times Time$

An estimate of the distance travelled would be

$$S'(t_1) \times 1 + S'(t_2) \times 1 + S'(t_3) \times 1.$$

Obviously sampling the speed more frequently will get more accurate results.



As we increase the number of samples, the thickness of these thin rectangles gets smaller and we end up with a more accurate result.

Definition of Definite Integrals

We define

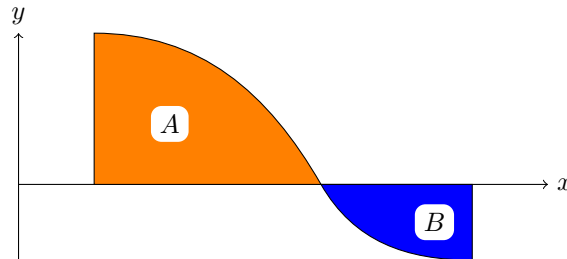
$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} S_n.$$

We call

$$\int_a^b f(x) dx$$

a *definite integral*. It is a number equal to the signed or algebraic area under the graph $y = f(x)$ between $x = a$ and $x = b$.

Negative Area



the definite integral gives:

$$\int_a^b f(x) dx = A - B$$

but the actual area is:

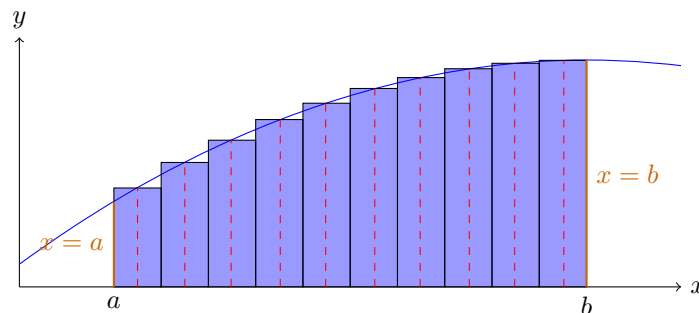
$$A + B = \int_a^b |f(x)| dx.$$

Estimating Definite Integrals - Riemann Sums

We can approximate the area under the graph of $y = f(x)$ between $x = a$ and $x = b$ by a *Riemann sum*

$$S_n = \sum_{i=1}^n f(t_i) \frac{b-a}{n},$$

$$a \leq t_1 \leq x_1 \leq t_2 \leq x_2 \leq \dots \leq x_{n-1} \leq t_n \leq b.$$



Estimating Definite Integrals

Example 1. A bird flies at a speed of $(5 - \frac{2}{3}t)$ km/h for two hours, from $t = 2$ to $t = 4$. Estimate the distance travelled.

First take 4 subdivisions

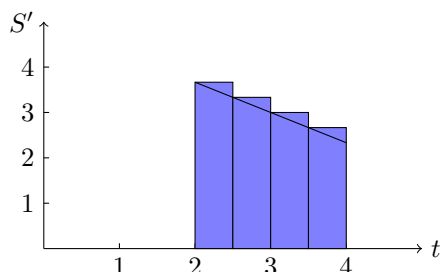
$$2 \leq 2.5 \leq 3 \leq 3.5 \leq 4$$

and take

$$t_1 = 2 \quad t_2 = 2.5 \quad t_3 = 3 \quad t_4 = 3.5.$$

Example 1 (continued). Estimate is

$$\begin{aligned} & \left(5 - \frac{2}{3} \times 2\right) \times 0.5 + \left(5 - \frac{2}{3} \times 2.5\right) \times 0.5 \\ & + \left(5 - \frac{2}{3} \times 3\right) \times 0.5 + \left(5 - \frac{2}{3} \times 3.5\right) \times 0.5 \\ & = 0.5\left[5 \times 4 - \frac{2}{3}(2 + 2.5 + 3 + 3.5)\right] \\ & = 6\frac{1}{3}. \end{aligned}$$



Example 1 (continued). Next take 5 subdivisions

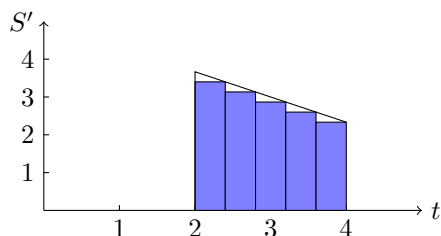
$$2 \leq 2.4 \leq 2.8 \leq 3.2 \leq 3.6 \leq 4$$

and take

$$t_1 = 2.4 \quad t_2 = 2.8 \quad t_3 = 3.2 \quad t_4 = 3.6 \quad t_5 = 4.$$

Example 1 (continued). Estimate is

$$\begin{aligned} & \left(5 - \frac{2}{3} \times 2.4\right) \times 0.4 + \left(5 - \frac{2}{3} \times 2.8\right) \times 0.4 \\ & + \left(5 - \frac{2}{3} \times 3.2\right) \times 0.4 + \left(5 - \frac{2}{3} \times 3.6\right) \times 0.4 \\ & + \left(5 - \frac{2}{3} \times 4\right) \times 0.4 \\ & = 0.4\left[5 \times 5 - \frac{2}{3}(2.4 + 2.8 + 3.2 + 3.6 + 4)\right] \\ & = 5\frac{11}{15}. \end{aligned}$$



Indefinite Integrals

Theorem 2 (Fundamental Theorem of Calculus). *The area under the curve $y = f(x)$ from $x = a$ to $x = b$ is*

$$\int_a^b f(x) dx = F(b) - F(a), \quad \text{where } F'(x) = f(x).$$

A function F such that $F' = f$ is called an

- antiderivative
- or a primitive
- or an indefinite integral.

The fundamental theorem of calculus tells us that we can evaluate $\int_a^b f(x) dx$ by finding an antiderivative $F(x)$ of $f(x)$:

$$\int_a^b f(x) dx = F(b) - F(a).$$

Example 3. The area under the curve $y = x^2$ between $x = 0$ and $x = 1$ is given by

$$\int_0^1 x^2 \cdot dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1^3}{3} - \left(\frac{0^3}{3} \right) = \frac{1}{3}.$$

Example 4. The slope of a curve at x is $2x + 1$ and the curve passes through $(1, 5)$. What is the equation of the curve?

We have

$$f'(x) = 2x + 1.$$

So

$$f(x) = \int (2x + 1) dx = x^2 + x + c.$$

Since $f(1) = 5$, we have $1 + 1 + c = 5$, i.e. $c = 3$.

So the curve is

$$y = x^2 + x + 3.$$

Example 5. Suppose that the rate of growth of a population is given by the differential equation

$$\frac{dP}{dt} = 100e^{\frac{1}{2}t},$$

where P is the number in the population at time t years after some starting date.

How much will the population increase in the first 10 years?

$$\begin{aligned} \text{Increase} &= P(10) - P(0) = [P(t)]_0^{10} \\ &= \int_0^{10} \frac{dP}{dt} dt \\ &= \int_0^{10} (100e^{\frac{1}{2}t}) dt \\ &= \left[200e^{\frac{1}{2}t} \right]_0^{10} = 200(e^5 - 1). \end{aligned}$$

Question 6. Suppose that the initial population in the previous example had been 200. Find a formula for the population at time t .