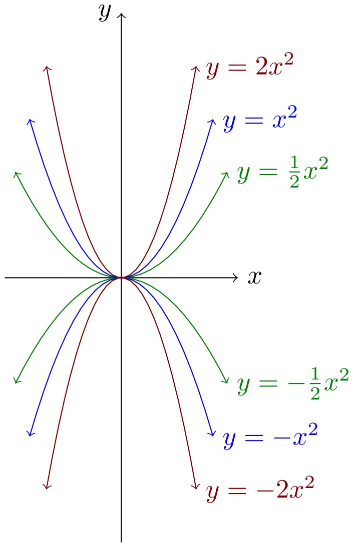
MATH1011 Lecture 2

Sinusoidal Functions

5 March 2010

Graphing Functions

Once we know how to plot a function, we can work out how to plot a number of variations of it.[6pt] You are already familiar with this for the basic quadratic function $y = x^2$.



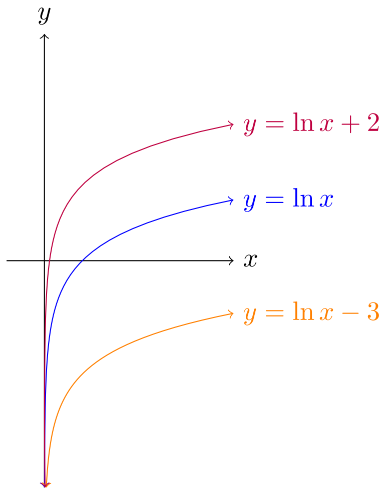
Fact 1.

- *Multiplying a function by a number* $a > 1$ *makes it "taller" or "skinnier".*
- *Multiplying a function by a number* $0 < a < 1$ *makes it "shorter" or "fatter".*
- *Multiplying a function by a number* $-1 < a < 0$ *makes it "shorter" or "fatter" and reflects it in the horizontal axis.*
- *Multiplying a function by a number* $a < -1$ *makes it "taller" or "skinnier" and reflects it in the horizontal axis.*

Vertical Shifts

Fact 2.

- *Adding a number to a function moves the graph of it up by that number.*
- *Subtracting a number from a function moves the graph of it down by that number.*



Fact 3. *Replacing x by $x + c$ moves the graph of $y = f(x)$ left c units.*

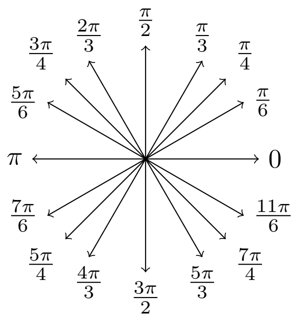
Fact 4. *Replacing x by $x - c$ moves the graph of $y = f(x)$ right c units.*

Beginnings of Trigonometry

We will use radians to measure angle, rather than degrees.

We always measure angles anticlockwise from the positive x-axis.

$$\pi \text{ radians} = 180^\circ.$$

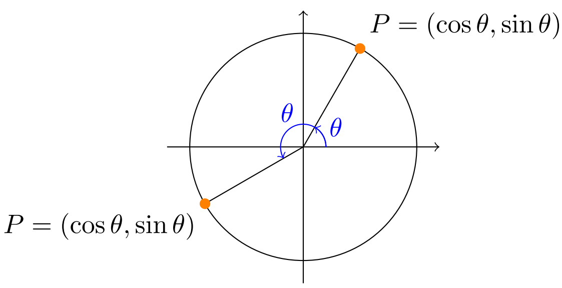


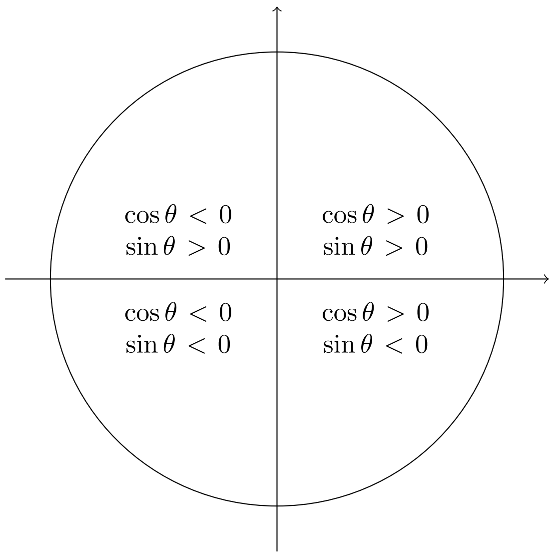
Unit Circle

We call the circle of radius 1 centred at the origin $(0, 0)$ the *unit circle*. We use the unit circle to define $\cos\theta$ and $\sin\theta$:

If P is a point on the unit circle, and θ the anticlockwise angle from the positive x-axis, then

- $\cos\theta = x$-coordinate of P;
- $\sin\theta = y$-coordinate of P.





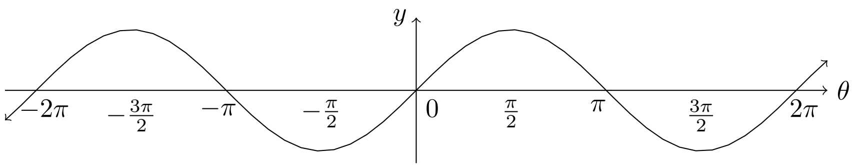
Sine and Cosine are Periodic functions

A periodic function satisfies $f(x+T) = f(x)$.

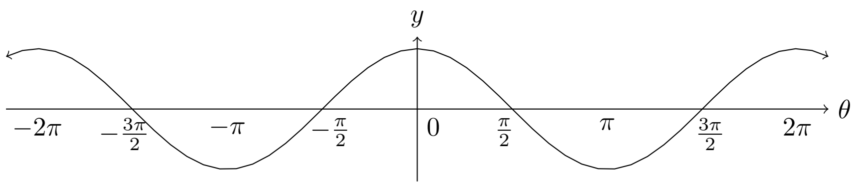
Many *periodic* phenomena are modelled by sinusoidal functions:

- alternating-current following along an electrical circuit
- pure sound waves formed by a tuning fork
- average daily temperature
- the height of an undamped spring

Using the unit circle, we see that the graph of $\sin\theta$ is:



whilst the graph of $\cos\theta$ is



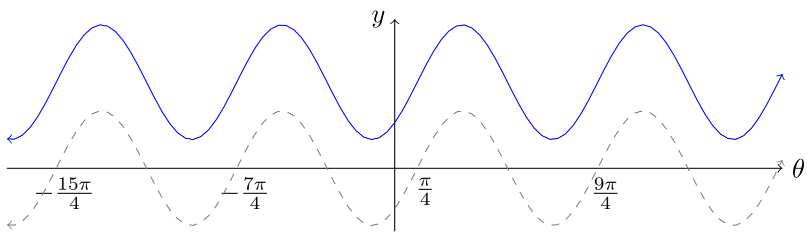
Notice that

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right),$$

and so we can express cosine in terms of sine.

Sketching Sinusoidal Functions

Example 5. Sketch the graph of $y = 3 + 2\sin\left(x - \frac{\pi}{4}\right)$.

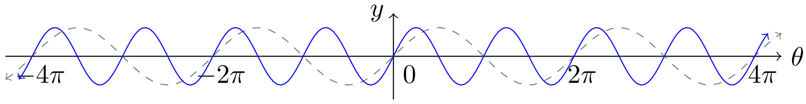


Begin with the graph of $\sin x$,
move it $\frac{\pi}{4}$ units to the right; stretch it vertically by a factor of 2; and move it up 3 units.

We can stretch/shrink sinusoidal functions *horizontally*.

Example 6. Sketch the graph of $y = \sin(2x)$.

The function $y = \sin u$ repeats every time u increases (or decreases) by 2π. Therefore it is *periodic* with *period* 2π.



If $u = 2x$, then u increases by 2π every time x increases by π. Hence the function $y = \sin(2x)$ is periodic with period π.

Vital Statistics

The general form of a sinusoidal function is

$$y = d + A \sin b(x - \alpha).$$

A, b, d and α are called vital statistics.

- A is called the *amplitude*;
- $p = \frac{2\pi}{b}$ is the *period*;
- d is the "mean value";
- α is the "starting point".

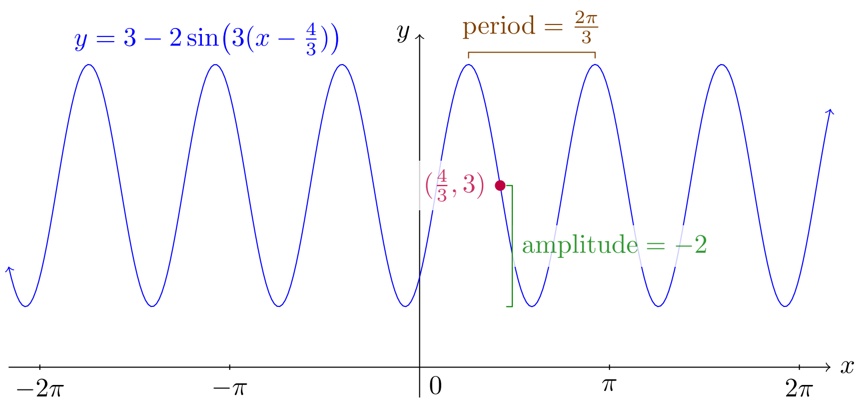
Example 7. Sketch the graph of the sinusoidal function:

$$\begin{aligned} y &= 3 - 2\sin(3x - 4) \\ &= 3 - 2\sin\left(3\left(x - \tfrac{4}{3}\right)\right) \end{aligned}$$

You need to factor out the coefficient of x in order to read off the vital statistics.

We have

- amplitude $= -2$ (the negative sign means that the graph is upside down);
- period $= \frac{2\pi}{3}$;
- mean value $= 3$; and
- start $= \frac{4}{3} \approx \frac{3}{7}\pi$.



Example 8. Find the equation of the following sinusoidal function:

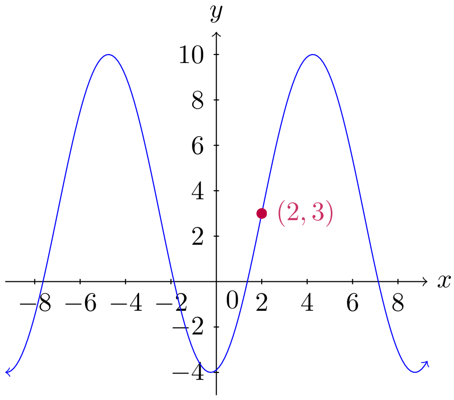
We estimate the vital statistics:

- amplitude 7;
- period 9;
- mean value 3;

- start 2;

Therefore the equation is:

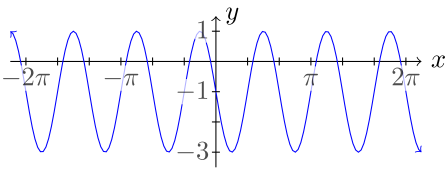
$$y = 3 + 7\sin\tfrac{2\pi}{9}(x-2).$$



Active Learning

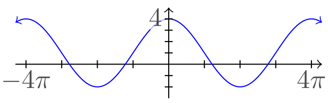
Question

1. Write down the equation of the function whose graph is

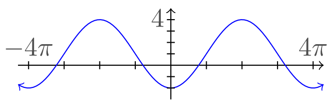


2. Which of these is the graph of $y = 3\sin\left(\frac{1}{2}(x+\pi)\right) + 1$?

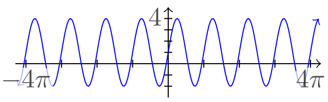
(a)



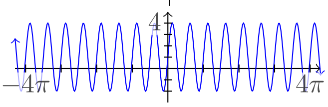
(b)



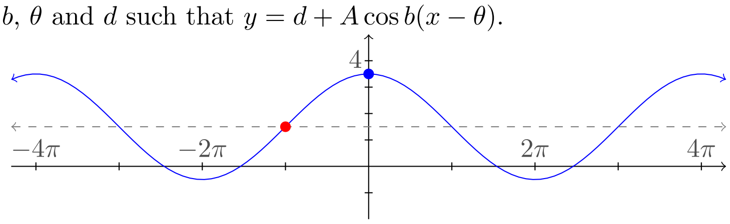
(c)



(d)



Example 9. Find A, b, θ and d such that $y = d + A\cos b(x-\theta)$.



From the graph we see that

- d = mean value = $\frac{3}{2}$
- A = amplitude = 2
- period = 4π, so $b = \frac{2\pi}{4\pi} = \frac{1}{2}$
- θ = start = 0

so

$$y = \frac{3}{2} + 2\cos\left(\frac{1}{2}x\right).$$

Combining sinusoidal functions having the same period

The sum of sinusoidal functions **with the same period**, can be expressed as a modified sine function,

$$a \sin x + b \cos x = R \sin(x + \alpha)$$

(where $R \geq 0$ and α are constants).

The problem is to find R and α, given a and b.

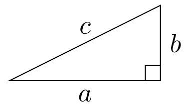
We need two trigonometric identities.

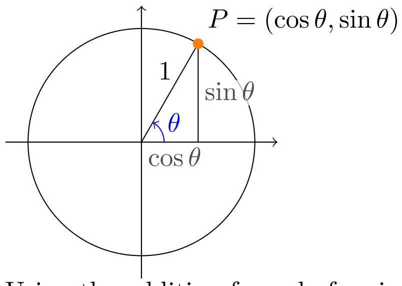
The first one is the addition formula for sine,

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

The second one comes from applying Pythagoras' theorem

$$a^2 + b^2 = c^2.$$





This gives the identity

$$\cos^2 \theta + \sin^2 \theta = 1.$$

Using the addition formula for sine,

$$\begin{aligned} R \sin(x + \alpha) &= R \sin x \cos \alpha + R \cos x \sin \alpha \\ &= (R \cos \alpha) \sin x + (R \sin \alpha) \cos x. \end{aligned}$$

If this is to equal $a \sin x + b \cos x$, then we must have

$$a = R \cos \alpha \qquad \text{and} \qquad b = R \sin \alpha.$$

Now we must solve for R and α in terms of a and b.

Squaring both equations and adding gives

$$\begin{aligned} a^2 + b^2 &= (R \cos \alpha)^2 + (R \sin \alpha)^2 \\ &= R^2 (\cos^2 \alpha + \sin^2 \alpha) = R^2 \times 1 \\ &= R^2, \end{aligned}$$

so that, as $R \geq 0$,

$$R = \sqrt{a^2 + b^2}.$$

Using $a = R \cos \alpha$ and $b = R \sin \alpha$ we have

$$\cos \alpha = \frac{a}{R} \quad \text{and} \quad \sin \alpha = \frac{b}{R}.$$

These equations determine a unique α in the range $0 \leq \alpha < 2\pi$.

Example 10. Graph

$$y = 4 \sin x - 3 \cos x + 1.$$

We begin by finding R and α such that

$$4 \sin x - 3 \cos x = R \sin(x + \alpha)$$

Using the addition formula as above,

$$4 \sin x - 3 \cos x = (R \cos \alpha) \sin x + (R \sin \alpha) \cos x,$$

so

$$R \cos \alpha = 4 \quad \text{and} \quad R \sin \alpha = -3.$$

Example 11.

$$R\cos\alpha = 4 \quad \text{and} \quad R\sin\alpha = -3.$$

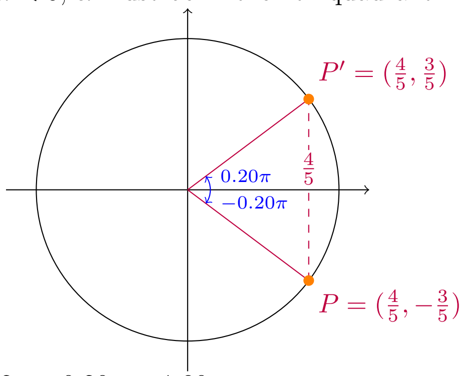
gives

$$R = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5,$$

$$\cos\alpha = \frac{4}{5} \quad \text{and} \quad \sin\alpha = -\frac{3}{5}.$$

Putting the first equation into your calculator may give the answer $\alpha \simeq 0.64 \simeq 0.20\pi$. Is this correct?

No! Since $\cos\alpha > 0$ and $\sin\alpha < 0$, α must be in the 4th quadrant.



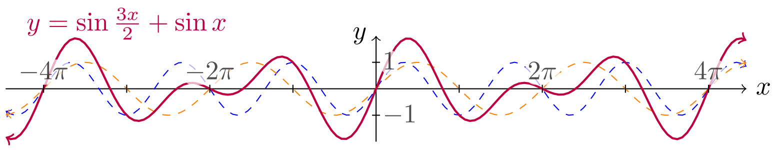
Instead, $\alpha \simeq -0.20\pi$ or $\alpha \simeq 2\pi - 0.20\pi = 1.80\pi$.

If two sinusoidal functions do not have the same period, then their sum is **not** in general sinusoidal. For example,

$$y = \sin\frac{3x}{2} + \sin x$$

is not sinusoidal, by which we mean that it cannot be written in the form

$$y = R\sin(x+\alpha) + d.$$



We see that the sum of sinusoidal functions is periodic. The period of the function above is 4π.