The University of Sydney

MATH1011 Applications of Calculus

Summer School	**Solutions to Exercise Sheet 10**	2012

Assumed knowledge:
Integration (definite and indefinite): basic functions, by recognition/substitution, and applications to rate problems. Basic understanding of the Fundamental Theorem of Calculus.

1. (*Preparatory Question*) An animal population is increasing at a rate of $100+40t+3t^2$ individuals per year (where t is measured in years). By how much does the animal population increase between the seventh and the tenth years?

Solution

The increase is given by the integral

$$\begin{aligned}\int_7^{10}(100+40t+3t^2)\,dt &= \left[100t+20t^2+t^3\right]_7^{10}\\ &= 100\cdot 10+20\cdot 10^2+10^3-100\cdot 7-20\cdot 7^2-7^3\\ &= 1977.\end{aligned}$$

2. Evaluate the definite integrals:

(*i*) $\displaystyle\int_0^{81}\left(\frac{2x}{27}+9x^3\sqrt{x}\right)dx$ (*ii*) $\displaystyle\int_1^{64}\frac{1+6t+15t^2}{\sqrt[3]{t}}\,dt$

Solution

(*i*)

$$\begin{aligned}\int_0^{81}\left(\frac{2x}{27}+9x^3\sqrt{x}\right)dx &= \frac{1}{27}\int_0^{81}2x\,dx+9\int_0^{81}x^{7/2}\,dx\\ &= \left[\frac{x^2}{27}+9\,\frac{x^{9/2}}{9/2}\right]_0^{81}\\ &= \left[\frac{x^2}{27}+2x^4\sqrt{x}\right]_0^{81}\\ &= \frac{1}{27}\,81^2+2\cdot 81^4\cdot\sqrt{81}\\ &= 774841221.\end{aligned}$$

(*ii*)

$$
\begin{aligned}
\int_1^{64} \frac{1+6t+15t^2}{\sqrt[3]{t}}\,dt &= \int_1^{64} t^{-1/3}\,dt + 6\int_1^{64} t^{2/3}\,dt + 15\int_1^{64} t^{5/3}\,dt \\
&= \left[\frac{t^{2/3}}{2/3} + 6\,\frac{t^{5/3}}{5/3} + 15\,\frac{t^{8/3}}{8/3}\right]_1^{64} \\
&= \left[\frac{3}{2}\sqrt[3]{t^2} + \frac{18}{5}\,t\sqrt[3]{t^2} + \frac{45}{8}\,t^2\,\sqrt[3]{t^2}\right]_1^{64} \\
&= \frac{3}{2}\sqrt[3]{64^2} + \frac{18}{5}\,64\cdot\sqrt[3]{64^2} + \frac{45}{8}\,64^2\cdot\sqrt[3]{64^2} \\
&\qquad - \frac{3}{2}\sqrt[3]{1^2} - \frac{18}{5}\,1\cdot\sqrt[3]{1^2} - \frac{45}{8}\,1^2\cdot\sqrt[3]{1^2} \\
&= 372339.675.
\end{aligned}
$$

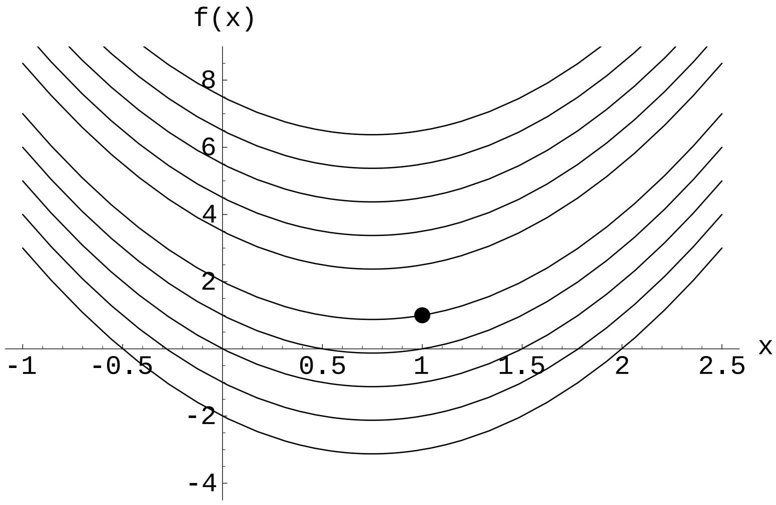
3. (*i*) Sketch the family of curves $y = f(x)$ which have slope equal to $f'(x) = 4x - 3$ at the point (x, y).

(*ii*) Find the equation of the particular curve which passes through the point $(1, 1)$.

Solution

(*i*) The curves with slope $(4x - 3)$ at the point (x, y) are the curves with equations of the form $y = 2x^2 - 3x + c$, where c is a constant, since $\dfrac{d}{dx}(2x^2 - 3x + c) = 4x - 3$, and no other functions have derivative $4x - 3$.

When $c = 0$ we have $y = 2x^2 - 3x$, the graph of which is an upright parabola passing through $(0, 0)$ and $(1\frac{1}{2}, 0)$. The family of curves consists of the parabolas obtained by shifting the curve $y = 2x^2 - 3x$ up or down.



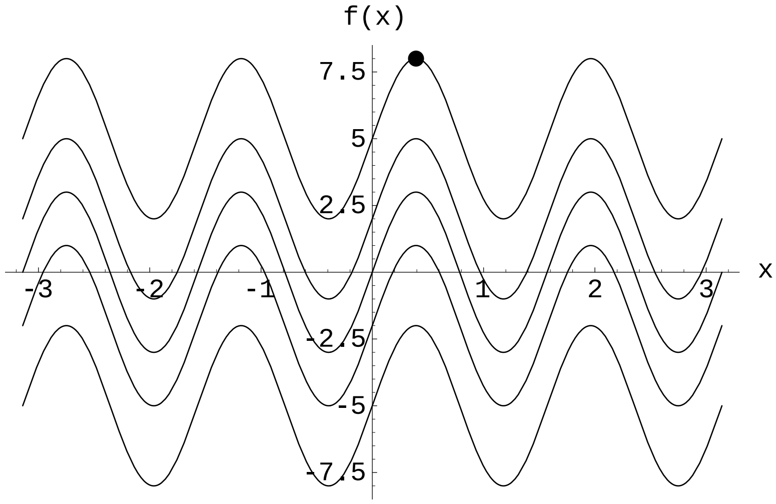
(*ii*) If the curve is to pass through the point $(1, 1)$, then we need $1 = 2 - 3 + c$. That is, $c = 2$. So the equation of the particular curve is $y = 2x^2 - 3x + 2$.

4. (*i*) Sketch a family of curves $y = f(t)$ which have slope equal to $f'(t) = 12\cos 4t$ at the point (t, y).

(*ii*) Find the equation of the particular curve which passes through the point $(\frac{\pi}{8}, 8)$.

Solution

(*i*) The slope at the point t equals the derivative $f'(t)$ of the function. So $f'(t) = 12\cos 4t$ and
$$f(t) = \int 12\cos 4t\,dt = 12 \cdot \frac{1}{4}\sin 4t + C = 3\sin 4t + C.$$
This is a periodic function with period $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$, amplitude 3 and mean value C. The family of curves looks as follows.



(*ii*) Substituting the coordinates of the point into the formula for $f(t)$ find the value for C:
$$8 = 3\sin 4\frac{\pi}{8} + C = 3 + C$$
that is, $C = 5$ and the equation is $y = 3\sin 4t + 5$.

5. Find the indefinite integrals:

(*i*) $\displaystyle\int \left(x - \frac{1}{x}\right)^2 dx$ (*ii*) $\displaystyle\int e^{5x-2}\,dx$ (*iii*) $\displaystyle\int \sin(3x+4)\,dx$

Solution

(*i*) $\displaystyle\int \left(x - \frac{1}{x}\right)^2 dx = \int \left(x^2 - 2 + \frac{1}{x^2}\right) dx = \frac{x^3}{3} - 2x - \frac{1}{x} + c.$

(*ii*) $\displaystyle\int e^{5x-2}\,dx = \tfrac{1}{5}\int 5e^{5x-2}\,dx = \frac{e^{5x-2}}{5} + c.$

(*iii*) $\displaystyle\int \sin(3x+4)\,dx = \tfrac{1}{3}\int 3\sin(3x+4)\,dx = \frac{-\cos(3x+4)}{3} + c.$

6. Find the indefinite integrals:

(*i*) $\displaystyle\int te^{-t^2}\,dt$

Hint: Differentiate $-t^2$ with respect to t.

(*ii*) $\displaystyle\int 6s\cos(3s^2+2)\,ds$

Hint: Differentiate $3s^2 + 2$ with respect to s.

Solution

(*i*) Use the formula

$$\int f(g(x))g'(x)dx = F(g(x)) + C,$$

where $F(t)$ is a primitive of $f(t)$.

Since $(-t^2)' = -2t$ we have

$$\int te^{-t^2}\, dt = -\frac{1}{2}\int (-2t)e^{-t^2}\, dt = -\frac{1}{2}\, e^{-t^2} + C.$$

(*ii*) Use the formula

$$\int f(g(x))g'(x)dx = F(g(x)) + C,$$

where $F(t)$ is a primitive of $f(t)$.

Since $(3s^2 + 2)' = 6s$ we have

$$\int 6s\cos(3s^2 + 2)\, ds = \sin(3s^2 + 2) + C.$$

7. Suppose that a person is injected with 40 mg of a certain drug, and that the drug is then continuously eliminated from the body, at the rate of $12.5e^{-0.06t}$ mg per hour after t hours, until it is eliminated completely.

(*i*) Let $A(t)$ be the number of mg of drug remaining in the body t hours after the drug was injected into the patient. What does the given information say about the derivative, $A'(t)$, of $A(t)$? (Note that $A(t)$ is *decreasing*.)

(*ii*) Find a formula for $A(t)$ in terms of t.

(*iii*) How much drug has been eliminated from the body after 1 hour?

(*iv*) After how many minutes will the person be drug-free?

Solution

(*i*) The rate of elimination of the drug is the negative of the derivative of $A(t)$, so

$$A'(t) = -12.5e^{-0.06t}.$$

(*ii*)

$$A(t) = \int (-12.5e^{-0.06t})\, dt = \frac{12.5}{0.06}\, e^{-0.06t} + C = 208.3e^{-0.06t} + C.$$

The amount of drug injected is the value of $A(t)$ at $t = 0$. So,

$$40 = 208.3 + C$$

which gives $C = -168.3$. Thus $A(t)$ is given by

$$A(t) = 208.3e^{-0.06t} - 168.3.$$

(*iii*) The amount of drug remaining in the body after 1 hour is

$$A(1) = 208.3e^{-0.06} - 168.3 \approx 27.9.$$

Therefore, $40 - 27.9 = 12.1$ mg of drug has been eliminated after 1 hour.

(*iv*) The drug will be eliminated completely when $A(t) = 0$, that is

$$208.3e^{-0.06t} - 168.3 = 0.$$

Solving this equation we find that

$$e^{-0.06t} = \frac{168.3}{208.3} \approx 0.8, \qquad -0.06\,t = \ln 0.8 \approx -0.21,$$

and so, $t \approx 3.55$ hours.

Thus the drug will be eliminated completely after approximately 213 minutes.

This can be estimated by looking at where the graph goes through the t-axis.

8. A cyclist travels at a speed of $\left(20 - \frac{t}{3}\right)$ kms/hr (after t hours).
How far does she travel in the first and second set of 6 hours?

Solution

Let the distance after t hours be $S(t)$. Then the speed after t hours is $S'(t) = 20 - \frac{t}{3}$.
In the first six hours, the distance travelled is

$$S(6) - S(0) = \int_0^6 \left(20 - \frac{t}{3}\right)\,dt = \left[20t - \frac{t^2}{6}\right]_0^6 = 114 \text{ km},$$

In the second six hours, the distance travelled is

$$S(12) - S(6) = \int_6^{12} \left(20 - \frac{t}{3}\right)\,dt = \left[20t - \frac{t^2}{6}\right]_6^{12} = 102 \text{ km}.$$

9. The rate of growth of a bacterial population, $P(t)$, is $10(e^t + 1)$ (t is measured in hours).

(*i*) By how much does the population increase in the first ten hours?

(*ii*) Find a formula for $P(t)$ if initially the population is 1000.

Solution

The rate of growth is the derivative of $P(t)$. Thus $\frac{dP}{dt} = 10(e^t + 1)$.

(*i*) In the first ten hours, the increase in the population is

$$\begin{aligned}\int_0^{10} \frac{dP}{dt}\,dt &= \int_0^{10} 10(e^t + 1)\,dt \\ &= 10[e^t + t]_0^{10} \\ &= 10(e^{10} + 9) \\ &\approx 220355.\end{aligned}$$

(*ii*)

$$P(t) = \int \frac{dP}{dt}\,dt = \int 10(e^t + 1)\,dt = 10(e^t + t) + c.$$

If $P = 1000$ when $t = 0$, then $1000 = P(0) = 10(e^0 + 0) + c = 10 + c$, and so $c = 990$.

Hence $P(t) = 10(e^t + t) + 990$.

Further practice questions from exercises in:

- *Stewart*
  - Basic integration
    - 5.3: 19-34, 69-74
    - 5.4: 5-12, 19-40
  - Integration with exponentials and logarithms (respectively)
    - 7.2: 73-82
    - 7.4: 69-80
  - Integration by substitution/recognition
    - 5.5: 1-30, 35-50,65-82
  - Rates of change and integration in the real world
    - 5.4: 47-60
- *Notes*
  - Definite integrals
    - Set 3.2: 1
  - Indefinite integrals using substitution/recognition
    - Set 3.3: 1
  - Integral curves
    - Set 3.3: 2
  - Rates of change and integration in the real world
    - Set 3.2: 4
    - Set 3.4: 1-4