The University of Sydney

MATH1011 Applications of Calculus

Summer School **Solutions to Exercise Sheet 9** 2012

Assumed knowledge:

Arithmetic and geometric progressions, limiting and telescoping sums. Applications of series to real world problems. Using series to estimate a definite integral in application to rate problems.

1. (*Preparatory Question*)

(i) Evaluate $\displaystyle\sum_{i=1}^{6} \frac{1}{3^{i-1}}$.

(ii) Find the sum of the first 50 even numbers (starting from the number 2).

(iii) Show[1] that $\dfrac{k}{(k+1)!} = \dfrac{1}{k!} - \dfrac{1}{(k+1)!}$. Hence find $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \cdots + \dfrac{29}{30!}$.

(iv) Write the sum of the first n terms of the sequence 5, 10, 20, 40, ... in sigma notation. Evaluate this sum for $n = 20$.

(v) Find $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3}{4}\right)^{k-1}$. What value of n gives $\displaystyle\sum_{k=1}^{n} \left(\frac{3}{4}\right)^{k-1} \approx 4$, correct to 3dp?

Solution

(i) Use the formula for geometric progressions:

$$\sum_{i=1}^{n} r^{i-1} = \frac{1-r^n}{1-r} \qquad \text{(for } a \neq 1.)$$

$$\begin{aligned}\sum_{i=1}^{6} \frac{1}{3^{i-1}} &= 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} \\ &= \frac{1-(\frac{1}{3})^6}{1-\frac{1}{3}} \\ &= \frac{3^6-1}{3^6-3^5} \\ &= \frac{728}{486} \\ &\approx 1.5.\end{aligned}$$

(ii) Use the formula for arithmetic progressions:

$$\begin{aligned}\sum_{i=1}^{n} (a+(i-1)d) &= \tfrac{1}{2}n\,(2a+(n-1)d) \\ &= \tfrac{1}{2}\,(\text{Number of Terms}) \times (\text{First Term} + \text{Last Term})\end{aligned}$$

$$2+4+6+\cdots+100 = \frac{1}{2} \times 50 \times (2+100) = 2550.$$

[1] Recall that $k! := k(k-1)(k-2)\ldots 3 \times 2 \times 1$. For example, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

(*iii*) This is a collapsing sum:

$$\begin{aligned}\frac{k}{(k+1)!} &= \frac{(k+1)-1}{(k+1)!} \\ &= \frac{k+1}{(k+1)!} - \frac{1}{(k+1)!} \\ &= \frac{k+1}{(k+1)k(k-1)\dots 3\cdot 2\cdot 1} - \frac{1}{(k+1)!} \\ &= \frac{1}{k!} - \frac{1}{(k+1)!}.\end{aligned}$$

The general term of the series is $\frac{k}{(k+1)!}$. So

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{29}{30!} = \sum_{k=1}^{29} \frac{k}{(k+1)!} = \sum_{k=1}^{29} \left[\frac{1}{k!} - \frac{1}{(k+1)!}\right].$$

So

$$\begin{aligned}\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{29}{30!} &= \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \dots + \left(\frac{1}{29!} - \frac{1}{30!}\right) \\ &= 1 - \frac{1}{30!}.\end{aligned}$$

The sum is slightly less than 1. However, $\frac{1}{30!}$ is so tiny that it does not even affect the 30th decimal place.

(*iv*) Each term is twice the previous one. Thus the terms are 5×2^0, 5×2^1, 5×2^2, and so on. Since the first term has 2^0, the second 2^1, the third 2^2, and so on, we see that the nth term is $5 \times 2^{n-1}$. Hence

$$5 + 10 + 20 + 40 + \dots + 5 \times 2^{n-1} = \sum_{k=1}^{n} 5 \times 2^{k-1}.$$

Using the formula for the sum of a geometric progression,

$$\sum_{k=1}^{20} 5 \times 2^{k-1} = 5\left(\frac{2^{20}-1}{2-1}\right) = 5 \times 1048575 = 5242875.$$

(*v*)

$$\sum_{k=1}^{n} \left(\tfrac{3}{4}\right)^{k-1} = \frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}} = 4\left(1-\left(\tfrac{3}{4}\right)^n\right).$$

Since $\left|\frac{3}{4}\right| < 1$ it follows that $\left(\frac{3}{4}\right)^n$ approaches 0 as n increases. So $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3}{4}\right)^{k-1} = 4.$

As we have seen, the true value of the sum is $4 - 4 \times \left(\frac{3}{4}\right)^n$, which differs from 4 by $4\left(\frac{3}{4}\right)^n$.
So if it is 4 to three decimal places, $4\left(\frac{3}{4}\right)^n$ has to be less than 0.0005. That is, we need

$$\left(\tfrac{3}{4}\right)^n < \frac{5}{40000},$$

or, equivalently,

$$\left(\tfrac{4}{3}\right)^n > \frac{40000}{5} = 8000.$$

Taking logs gives $n \ln\left(\frac{4}{3}\right) > \ln 8000$, or $n \times 0.2877 > 8.9872$. It follows that we need $n > \dfrac{8.9872}{0.2877} \approx 31.2$. Using a calculator we find that

$$\sum_{i=1}^{31} (\tfrac{3}{4})^{i-1} \approx 3.99946425373, \qquad \sum_{i=1}^{32} (\tfrac{3}{4})^{i-1} \approx 3.9995981903.$$

2. Show that[2] $k.k! = (k+1)! - k!$. Hence find $\displaystyle\sum_{k=1}^{50} k.k!$.

Solution

This is another collapsing sum!

$$(k+1)! - k! = (k+1)k! - k! = k!(k+1-1) = k.k!.$$

$$\begin{aligned}\sum_{k=1}^{50} k.k! &= \sum_{k=1}^{50}((k+1)! - k!) \\ &= (2! - 1!) + (3! - 2!) + (4! - 3!) + \cdots + (50! - 49!) + (51! - 50!) \\ &= 51! - 1.\end{aligned}$$

[2]These "show that" questions are a bit tricky, so just assume they are true for now if you can't show it!

3. Suppose that you invest \$10000 in an annuity at the beginning of 1999, and that you add \$2000 to the investment at the beginning of each successive year. The interest rate is 5.5% per annum, and interest is added annually, at the end of each year. How much money will you have at the end of the year 2005 (after the interest for that year has been added)?

Solution

Let $\$A_n$ be the amount at the end of the nth year after 1998 (so that $\$A_1$ is the amount at the end of 1999, and $\$A_7$ the amount at the end of 2005). Then

$$A_1 = 10000 \times 1.055$$

$$\begin{aligned} A_2 &= (A_1 + 2000) \times 1.055 \\ &= (10000 \times 1.055 + 2000) \times 1.055 \\ &= 10000 \times 1.055^2 + 2000 \times 1.055 \end{aligned}$$

$$\begin{aligned} A_3 &= (A_2 + 2000) \times 1.055 \\ &= 10000 \times 1.055^3 + 2000 \times 1.055^2 + 2000 \times 1.055 \end{aligned}$$

$$\vdots$$

$$\begin{aligned} A_7 &= 10000 \times 1.055^7 + 2000 \times 1.055^6 + 2000 \times 1.055^5 + \cdots + 20000 \times 1.055 \\ &= 10000 \times 1.055^7 + 2000 \times 1.055(1.055^5 + 1.055^4 + \cdots + 1.055 + 1) \\ &= 10000 \times 1.055^7 + 2000 \times 1.055 \times \frac{1.055^6 - 1}{1.055 - 1} \\ &\approx 29080. \end{aligned}$$

At the end of the year 2005 there will be \$29080 in the annuity.

4. A person makes a one-off investment of \$10000 to which interest is added half-yearly at the rate of 8% per annum. The principal is to accumulate for ten years and then a regular annuity is to be paid in ten equal yearly instalments after which the principal will have been reduced to zero. The first instalment is to be paid immediately after the twentieth half-yearly interest payment is made. How much will the investor receive in each instalment?

Solution

After ten years, twenty interest payments at 4% will have been made, and the amount of money accumulated will be $P = 10000 \times 1.04^{20} \approx 21911$. Now let $\$A$ be the amount of each yearly instalment (paid at the beginning of each year).

Then the amount remaining at the beginning of year 11 (after the first instalment is paid) is $P - A$, and during year 11 this earns 4% interest each six months. So at the beginning of year 12, after the next instalment is paid, the amount will be $(P-A)\times 1.04^2 - A$.Similarly, at the beginning of year 13, after the third instalment is paid,the amount will be

$$((P-A)\times 1.04^2 - A)\times 1.04^2 - A = P\times 1.04^4 - A\times 1.04^4 - A\times 1.04^2 - A.$$

Continuing like this, we see that at the beginning of year 20, after the final instalment has been paid, the amount left will be

$$\begin{aligned} P\times 1.04^{18} - A\times 1.04^{18} - A\times 1.04^{16} - \ldots A\times 1.04^2 - A \\ = P\times 1.04^{18} - A(1.04^{18} + 1.04^{16} + \cdots + 1.04^2 + 1) \\ = P\times 1.04^{18} - A\times \frac{(1.04^2)^{10} - 1}{1.04^2 - 1}. \end{aligned}$$

However, this amount must be zero, and hence

$$P\times 1.04^{18} = A\times \frac{(1.04^2)^{10} - 1}{1.04^2 - 1}.$$

But $P = 10000\times 1.04^{20}$; so

$$A = \frac{10000\times 1.04^{20}\times 1.04^{18}\times (1.04^2 - 1)}{1.04^{20} - 1} \approx 3040.$$

The investor therefore receives \$3040 each year.

5. After t hours, an organism is producing cells at the rate of $20e^t$ cells per hour.

(i) Estimate the number of cells, $N(t)$, produced in the first four hours by dividing the interval $0 \le t \le 4$ into eight equal sub-intervals and evaluating the sum $\sum_{i=1}^{8} N'(t_i) \times \frac{4-0}{8}$ (where each t_i is in the ith sub-interval).

(ii) Find the number of cells produced in the first 4 hours by evaluating a definite integral.

Solution

(i) If the number of cells is $N(t)$, and this is changing at the rate of $20e^t$ cells per hour, then $N'(t) = 20e^t$.
The sub-intervals are $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[1, 1\frac{1}{2}]$, $[1\frac{1}{2}, 2]$, $[2, 2\frac{1}{2}]$, $[2\frac{1}{2}, 3]$, $[3, 3\frac{1}{2}]$ and $[3\frac{1}{2}, 4]$. If we choose t_i in the middle of the ith sub-interval, then $t_1 = \frac{1}{4}$, $t_2 = \frac{3}{4}$, $t_3 = 1\frac{1}{4}$, $t_4 = 1\frac{3}{4}$, $t_5 = 2\frac{1}{4}$, $t_6 = 2\frac{3}{4}$, $t_7 = 3\frac{1}{4}$ and $t_8 = 3\frac{3}{4}$. So

$$\begin{aligned} &\sum_{i=1}^{8} N'(t_i)\times \tfrac{1}{2} = \tfrac{1}{2}\sum_{i=1}^{8} N'(t_i) = \tfrac{1}{2}\sum_{i=1}^{8} 20e^{t_i} = 10\sum_{i=1}^{8} e^{t_i} \\ &= 10(e^{0.25} + e^{0.75} + e^{1.25} + e^{1.75} + e^{2.25} + e^{2.75} + e^{3.25} + e^{3.75}) \\ &\approx 1060. \end{aligned}$$

Our estimate for the number of cells produced in the first 4 hours is 1060. (You may, of course, choose other values for t_i, so long as each value lies in the ith sub-interval. Your estimate will then be slightly different from the one above.)

(*ii*) The number of cells produced in first 4 hours is $N(4)-N(0)$. Expressed as a definite integral, this is

$$\int_0^4 N'(t)\,dt = \int_0^4 20e^t\,dt = \left[20e^t\right]_0^4 = 20e^4 - 20e^0 \approx 1072.$$

(Alternatively, since $N'(t) = 20e^t$ it follows that $N(t) = 20e^t + C$ for some constant C, whence $N(4) - N(0) = 20e^4 - 20e^0$.)
The fact that we were able to find a function whose derivative is equal to $20e^t$ saved us from having to attempt a direct computation of $\lim_{n\to\infty} \sum_{i=1}^{n} N'(t_i) \times \frac{4-0}{n}$.

Further practice questions from exercises in:

- *Stewart*
  - Applications of series to the real world[3]
    - 12.2: 58, 64
  - Approximating integration using series[4]
    - 5.2: 1-4, 7-8
- *Notes*
  - Applications of series to the real world
    - Set 3.1: 3-4
  - Rates of change and approximating integration using series
    - Set 3.2: 2-3

[3]These problems are challenging!
[4]Referred to by *Stewart* as *Riemann Sums*. Beware these questions are slightly more difficult than what we've done in class, but useful as extra practice.